Safety, in Numbers

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Abstract

We introduce a way to compare actions in decision problems. An action is *safer* than another if the set of beliefs at which the decision-maker prefers the safer action increases in size (in the set-inclusion sense) as the decision-maker becomes more risk averse. We provide a full characterization of this relation and show that it is equivalent to a robust concept of single-crossing. We discuss applications to investment hedging, security design, and game theory.

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1 Introduction

Risk is a central component of decision making under uncertainty. By now it is understood that in many situations, decision-makers (DMs) dislike risk, and such

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aversion affects a DM's behavior when she makes some choice. Moreover, there is also a standard notion of what it means for one DM to be more risk averse than another: in the expected-utility realm, the DM that is more risk averse has a utility that is a strictly monotone concave transformation of the other's.

It is less well-understood, however, what characteristics lead some actions to be more attractive to DMs than others. Casual reflection suggests there is at least a vague notion of an action (a choice the DM can take) being less risky than another. For example, if the decision problem is a choice of bet at a roulette table, betting on *red* is safer than betting on a specific odd number; and leaving the table, money in hand, is safer than either.

Our goal in this paper is to understand and define precisely what it means for one action in a decision problem to be safer than another. Our criterion is a broad one: we fix an arbitrary decision problem–a triple consisting of a set of states, a set of actions, and a state-and-action-dependent utility function–and formulate a binary relation between actions available to the DM. One action a is Safer than another action b, $a \succeq_S b$, if the set of beliefs at which action a is preferred to b grows–in the set inclusion sense–as the DM becomes more risk averse. That is, safer actions become more attractive as the DM becomes more risk averse.

What makes one action safer than another? It is useful to first think about a decision problem with a risk-free action, which yields the same payoff to the DM no matter the state. In the roulette example, this is leaving the table, money in hand. As we note above, we would expect such an action to be safer than any other action: increased risk aversion *should* make the risk-free action relatively more attractive than one that exposes the agent to risk.[1] In belief space, the (expected) payoff of a risk-free action has a completely flat slope, so a natural guess is that safer actions are those with flatter slopes.

[1] That a less risk-averse agent accepts any gamble that a more-risk averse agent does is precisely Yaari (1969)'s behavioral definition of risk aversion.

Our main result, Theorem 4.1, establishes necessary and sufficient conditions on the DM's payoffs for action a to be safer than action b. Surprisingly, even with just two states, having a flatter slope is too weak–our "safer than" relation implies a flatter slope but the converse is not true. Specifically, a is safer than b if and only if the payoffs to a lie within the convex hull of the payoffs to b. Note that *this condition relies on payoffs alone*; it is independent of the DM's belief.

We prove that when there are more than two states, the conditions equivalent to safety are simply the aggregation of a particular collection of two-state conditions. Namely, it is sufficient to examine only the edges of the simplex of beliefs that contain points of indifference between the two actions. We treat each such edge like a two-state environment and so the two-state safer-than conditions (for each relevant edge) are all that we need.

Connection to "Single-Crossing:" Naturally, for any fixed distribution over states x, any choice of action a yields a real-valued random variable (in utils) with distribution F_a^x. Working with these objects, there is a classical notion of the robustness with respect to increased risk aversion of preferring one action to another. As implied by results in Karlin and Novikoff (1963), if F_a^x crosses F_b^x once from below, then $\mathbb{E}_{F_a^x}[u] \geq \mathbb{E}_{F_b^x}[u]$ implies $\mathbb{E}_{F_a^x}[\phi \circ u] \geq \mathbb{E}_{F_b^x}[\phi \circ u]$ for any strictly increasing concave ϕ–any increase in risk-aversion.

Crucially, this classical notion is distribution-specific: *single-crossing is a property of distributions over payoffs.* In contrast, *our concept is a statement about the payoffs themselves.* Nevertheless, we show that these two properties are intimately connected. In Theorem 4.7, we establish that a is safer than b if and only if F_a^x single-crosses F_b^x from below for any distribution over states x.

Applications: We discuss the usefulness of the safer-than relation in several applications. First, we relate our relation to hedging in an investment setting. Given an investor's current holdings, we formulate a binary relation between assets: one asset "hedges" better than another if the set of beliefs justifying it expands as the investor becomes more risk averse. Second, we use our relation to rank securities according to their sensitivity to an investor's risk aversion. Third, we briefly discuss "safe" strategy profiles in coordination games, which is similar in spirit, but not equivalent, to the risk-dominance concept of Harsanyi and Selten (1988).

Proof Approach: In deducing the safer-than relation, we begin with the two-state environment: there, beliefs are scalars and regions of optimality for beliefs are intervals, so the proof requires only characterizing in which direction the point of indifference between the two actions moves as a result of the DM's increased risk aversion. To do this, we use an elementary result from convex analysis–the three-chord lemma. Unanticipatedly, it proves straightforward to extend the characterization of the relation to a general state space. We discover that the general case can be understood as a collection of two-state environments, and so the necessary and sufficient condition of the theorem is just that the two-state conditions hold for every pair in the collection.

The intuition behind this–that all we need to do is aggregate the two-state conditions–is as follows. We require the set of optimal beliefs for an action to grow, *viz.*, the set of beliefs at which the specified action is initially optimal must be a subset of the new set of beliefs after the DM becomes more risk averse. These optimality sets are simply the intersections of half-spaces and the probability simplex. Moreover, the extreme points of these sets are vertices of the simplex and certain non-vertex points on the edges. These points on the edges are precisely the indifference points between the actions for pairs of states in which each of the two actions is optimal in only one state. We need only make sure these indifference points move in the "right direction" along the edges as the DM becomes more risk averse.

1.1 Related Work

Yaari (1969) introduces an important comparative notion of risk aversion: "Mr. A is more risk averse than Mr. B if...every gamble which is acceptable to A is acceptable to B." In other words, if A is more risk averse than B, the set of risky actions that he prefers to a risk-free action is a subset of the risky actions that B prefers to the risk-free one.[2] Using our concept of safety, an action that has a state-independent (deterministic) payoff is safer than all other actions. Moreover, we argue that there is a natural way to broaden this notion of insensitivity to risk-aversion beyond a total absence of risk. That is, Yaari's behavioral characterization of "less risk averse" can be redefined to mean increased willingness to pick an action over a safer one, instead of just the safest one.

Hammond III (1974), Lambert and Hey (1979), Karlin and Novikoff (1963), Jewitt (1987) all contain results concerning when an action that is preferred to another given some utility function must still be preferred following any increase in the agent's risk aversion. The key difference is that the condition in these papers is on the *distributions* over wealth obtained by the agent as a result of her choice of actions. In this context, one contribution of our paper is to formulate a way of comparing actions' comparative robustness that is *distribution-free*. In §4.2, we connect our concept of safety to the single-crossing property identified by these works and reveal that safety is equivalent to a robust notion of single-crossing of the induced distributions over utils for all subjective beliefs.

Whitmeyer (2023) is also related to our work. That paper studies transformations of decision problems that render information more valuable to a DM. Here, we study a particular variety of transformation–an increase in the DM's risk aversion–and focus on its effect on the optimality of various actions.

This paper also harkens to the comparative statics literature; see, e.g., Milgrom

[2] Similarly, Ghirardato and Marinacci (2002) defines when preferences are "more ambiguity averse" than others.

and Shannon (1994), Edlin and Shannon (1998), and Athey (2002). We also vary a parameter, the DM's risk aversion, and ask how this affects the DM's behavior. However, we focus on comparisons between actions and make our relation quite demanding: the enlargement of the set of beliefs at which an action is preferred to another must arise for any monotone concave transformation of the DM's utility.

Finally, our work is also related to the striking result of Battigalli, Cerreia-Vioglio, Maccheroni, and Marinacci (2016), who prove that increased risk aversion on the part of the DM enlarges the set of justifiable actions, the actions that are at least weakly optimal at some belief. That is, a justifiable action remains justifiable if the DM becomes more risk averse; and increased risk aversion may render actions optimal that had previously been strictly dominated.[3] We, instead, study the properties of decision problems and actions therein for which increased risk aversion enlarges the set of beliefs at which some actions are optimal.

2 Model

There is an unknown state of the world θ, which is an element of some topological space of states Θ, endowed with the Borel σ-algebra. We specify further that Θ is compact and metrizable. We denote the set of all Borel probability measures on Θ by $\Delta \equiv \Delta(\Theta)$. Our protagonist is a decision-maker (DM) with a compact set of actions $A \subseteq \mathbb{R}$ (with $|A| \geq 2$), and a state dependent utility function $u: A \times \Theta \to \mathbb{R}_+$.[4]

We assume u is continuous in the action, a, the DM has a subjective belief $x \in \Delta$, and she is a subjective expected-utility (EU) maximizer. We also specify that no action in A is weakly dominated: for all $a \in A$, there exists some $x \in \Delta$ at which a is uniquely optimal. The DM becomes more risk averse if her utility function is

[3]By understanding a decision problem as a game with just a single player, this result is also shown by Weinstein (2016).

[4]We posit a "reduced-form" utility mapping actions and states to utils. We could just as easily specify a set of consequences and (Savage) acts, but prefer the current form.

instead $\hat{u}$ where $\hat{u} = \phi \circ u$ for some strictly monotone concave ϕ.

For any two actions $a, b \in A$ $(a \neq b)$, we define the set $P_{a,b}(a)$ to be the subset of the probability simplex on which action a is weakly preferred to b; formally,

$$P_{a,b}(a) := \{x \in \Delta \colon \mathbb{E}_x u(a,\theta) \geq \mathbb{E}_x u(b,\theta)\}.$$

By assumption this set is non-empty and–when Δ is finite dimensional–of full dimension in Δ. When the utility function is $\hat{u}$, we define the set $\hat{P}_{a,b}(a)$ in the analogous manner.

Definition 2.1. Action a is Safer than action b if for any monotone concave ϕ, $P_{a,b}(a) \subseteq \hat{P}_{a,b}(a)$; i.e., the set of beliefs at which action a is preferred to b increases in size as the DM becomes more risk averse.

Equivalently, action a is safer than action b if

$$\mathbb{E}_x u(a,\theta) \geq \mathbb{E}_x u(b,\theta) \;\Rightarrow\; \mathbb{E}_x \phi \circ u(a,\theta) \geq \mathbb{E}_x \phi \circ u(b,\theta)$$

for any strictly monotone concave ϕ. Let $a \succeq_S b$ denote the binary relation Action a is safer than Action b. The strict relation, $a \succ_S b$ denotes $a \succeq_S b$ but $b \not\succeq_S a$.

3 Two States

When there are two states, 0 and 1, for two actions $a, b \in A$, let $\alpha_\theta \equiv u(a,\theta)$ and $\beta_\theta \equiv u(b,\theta)$. As no action is weakly dominated, and as we could just relabel the actions, we specify without loss of generality that $\alpha_0 > \beta_0$ and $\beta_1 > \alpha_1$ so that if the DM knows that the state is 0, she prefers a and if she knows that the state is 1, she prefers b.

3.1 Safer Actions

Consider the DM's choice between actions a and b. Let $x \in [0,1]$ be the probability that the state is 1, so that she prefers action a if and only if

$$(1-x)\alpha_0 + x\alpha_1 \geq (1-x)\beta_0 + x\beta_1 \;\Leftrightarrow\; x \leq \frac{\alpha_0 - \beta_0}{\alpha_0 - \beta_0 + \beta_1 - \beta_0} \equiv \bar{x}.$$

Analogously, under the transformed utility, the indifference belief is

$$\hat{x} \equiv \frac{\phi(\alpha_0) - \phi(\beta_0)}{\phi(\alpha_0) - \phi(\beta_0) + \phi(\beta_1) - \phi(\alpha_1)}.$$

By definition, action a is safer than action b if it is chosen for more beliefs after the change in utility; that is, if $\bar{x} \leq \hat{x}$. This translates to

$$\frac{\phi(\beta_1) - \phi(\alpha_1)}{\beta_1 - \alpha_1} \leq \frac{\phi(\alpha_0) - \phi(\beta_0)}{\alpha_0 - \beta_0}. \tag{1}$$

Consider the meaning of this condition. The right-hand side is the secant line to the concave ϕ at points α_0 and β_0; that is, the average slope of ϕ between β_0 and α_0. In other words, it is the marginal benefit from "correctly" choosing a in state 0 (because a is better if the state is 0) under the transformation ϕ. Therefore, if action a is safer than action b, its marginal benefit in its "correct" state is higher under the transformation, and it will be chosen more.

Using Inequality (1), we can characterize the "safer than" binary relation, $\succeq_S$, exactly when there are two states:

Proposition 3.1. *For actions $a, b \in A$, $a \succeq_S b$ if and only if $\beta_1 \geq \alpha_0$ & $\alpha_1 \geq \beta_0$.*

Proof. ($\Rightarrow$) By the Three-chord lemma (Theorem 1.16 in Phelps (2009))

$$\frac{\phi(\alpha_0) - \phi(\beta_0)}{\alpha_0 - \beta_0} \geq \frac{\phi(\beta_1) - \phi(\beta_0)}{\beta_1 - \beta_0}. \tag{2}$$

Likewise, $\beta_1 > \alpha_1 \geq \beta_0$ plus the Three-chord lemma imply

$$\frac{\phi(\beta_1) - \phi(\beta_0)}{\beta_1 - \beta_0} \geq \frac{\phi(\beta_1) - \phi(\alpha_1)}{\beta_1 - \alpha_1}. \tag{3}$$

Combining Inequalities 2 and 3 yields Inequality 1.

($\Leftarrow$) See Appendix A.1. ∎

Recalling that $\alpha_0 > \beta_0$ and $\beta_1 > \alpha_1$, we can rephrase this proposition as saying that a is safer than b if and only if the payoffs of a lie in the convex hull of the payoffs of b. In other words, $a \succeq_S b$ if and only if choosing a and being "wrong" (because the state is actually 1) is not as bad as choosing b and being "wrong," while choosing a and being "right" is not as good as choosing b and being "right."

One might suppose that actions that yield comparatively consistent payoffs are relatively safe actions. In other words, if we let the slope of the payoff to action a be $\gamma_a := \alpha_1 - \alpha_0$ and the payoff to action b be $\gamma_b := \beta_1 - \beta_0$, then a reasonable guess is that an action with a shallower slope, or smaller (absolute value of) γ_i, is safer. This is not always the case, however.

Corollary 3.2. *If $a \succeq_S b$, then $\left|\gamma_a\right| \leq \left|\gamma_b\right|$. The converse is not generally true.*

Proof. See Appendix A.2. ∎

To gain intuition for why a shallower slope of the payoff function is not sufficient for a safer action, consider the following example. Let $\alpha_0 = 5$, $\beta_1 = 4$, $\alpha_1 = 3$, and $\beta_0 = 1$. Then, $|\gamma_a| = 2 < 3 = |\gamma_b|$ so that action a has a shallower slope, but $\beta_1 < \alpha_0$ so that a is not safer than b.

Now define $W(x) \equiv \max_{a' \in \{a,b\}} \mathbb{E}_x u(a', \theta)$. It is only if the slope of W changes sign that we lose the equivalence between safety and "flatter than."[5]

Corollary 3.3. *If $W(x)$ is monotone, then $\left|\gamma_a\right| \leq \left|\gamma_b\right| \Rightarrow a \succeq_S b$.*

[5] In Appendix B we present an example with a quadratic loss utility function demonstrating Proposition 3.1 and Corollary 3.3.

Proof. WLOG, let W be increasing. Then, $\beta_1 > \alpha_1 \geq \alpha_0$ and $\alpha_1 \geq \alpha_0 > \beta_0$. ■

For any action a', the DM's expected payoff is $(1-x)u(a',0)+xu(a',1)$ with slope $u(a',1) - u(a',0)$. W being monotone means that the DM prefers one state to the other no matter the action so that the closer she gets to believing that the preferred state is the true state, the better off she is.[6]

4 More Than Two States

Now let Θ be an arbitrary compact and metrizable space (endowed with the Borel σ-algebra). We restrict attention to generic decision problems, in which the DM strictly prefers one of the two actions in each state.[7] For two actions $a, b \in A$ we maintain the notation $\alpha_\theta := u(a,\theta)$ and $\beta_\theta := u(b,\theta)$. We maintain the assumption that no action is weakly dominated, and define $\mathcal{A}$ to be the set of states in which a is uniquely optimal and $\mathcal{B} = \Theta \setminus \mathcal{A}$ to be the set of states in which b is uniquely optimal.

4.1 Safer Actions

Now, we state the main result of the paper.

Theorem 4.1. *Action a is safer than action b if and only if for each $\theta \in \mathcal{A}$ and $\theta' \in \mathcal{B}$, $\beta_{\theta'} \geq \alpha_\theta$ and $\alpha_{\theta'} \geq \beta_\theta$.*

Proof. See Appendix A.3. ■

The key to this result is that we need only compare action by action and state by state along particular edges of Δ. Specifically, consider states $\theta \in \mathcal{A}$ and $\theta' \in$

[6] We give conditions for the action set to be totally ordered with respect to safety for the general case in Proposition 4.8 below.

[7] This assumption is innocuous, allowing us to save on notation and work while leaving the results unchanged.

$\mathcal{B}$. Comparing payoffs for only these two states, our result from Proposition 3.1 applies directly so that, if the probability of all other states is 0, a is safer than b if and only if $\alpha_\theta \le \beta_{\theta'}$ and $\beta_\theta \le \alpha_{\theta'}$, as before. Doing this for every pair of states in which $\theta \in \mathcal{A}$ and $\theta' \in \mathcal{B}$ yields the result.

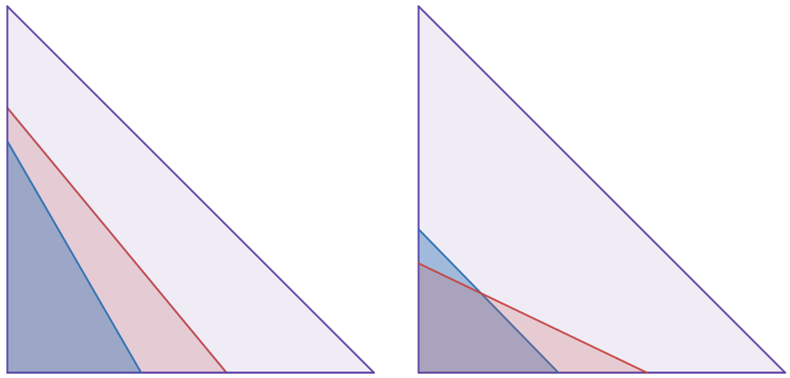
Figure 1 illustrates two safety comparisons when there are three states. In both panels, the blue region is the set of beliefs at which action a is preferred to action b when the DM's utility is u, and the red region is the set of beliefs at which a is preferred to b when the DM's utility is $\hat{u} = \phi \circ u$, where $\phi(\cdot) = (\cdot)^{\frac{1}{t}}$. In the left panel, $a \succeq_S b$, so that when the DM becomes more risk averse, the set of beliefs at which a is preferred expands. In the right panel, however, $\alpha_2 < \beta_0$, violating Theorem 4.1's conditions.

Some decision problems contain a risk-free action, i.e., one that guarantees a deterministic payoff to the DM. Formally, action a is risk-free if $u(a,\theta) = u(a,\theta')$ for all $\theta, \theta' \in \Theta$. Keep in mind that our specification that no action is weakly dominated implies that there is at most one risk-free action. Risk-free actions interact with our relation in a natural way.

Corollary 4.2. *If there exists a risk-free action, a, then $a \succ_S b$ for all $b \neq a$.*

4.2 Connection to "Single-Crossing"

Decision-making under uncertainty has long been of interest to economists. A classical question concerns which lotteries are made comparatively more attractive as agents become more risk averse. This is similar in spirit to our safety notion, though as we will shortly explain, there is an important difference: the classical relation is between lotteries, which are random variables, whereas ours is between vectors of (state-dependent) payoffs.



(a) (Interactive Version) (b) (Interactive Version)

Figure 1: In both panels, there are three possible states: 0, 1, and 2. The x axis represents the probability that the state is 1, while the y axis represents the probability that the state is 2. The blue areas represent beliefs at which a is preferred to b under the original utility specification, while the red areas indicate beliefs at which a is preferred to b after the concave transformation. In both panels, $0 \in \mathcal{A}$, $1, 2 \in \mathcal{B}$, $\alpha_0 \leq \beta_2$, $\alpha_0 \leq \beta_1$, and $\beta_0 \leq \alpha_1$. In the left panel, $\beta_0 \leq \alpha_2$ ($a \succeq_S b$), while in the right panel, $\alpha_2 < \beta_0$ ($a \not\succeq_S b$ and $b \not\succeq_S a$).

Nevertheless, we can connect these two concepts. Given belief $x \in \Delta$, let

$$F_a^x(v) := \mathbb{P}_x(u(a,\theta) \leq v), \quad \text{and} \quad F_b^x(v) := \mathbb{P}_x(u(b,\theta) \leq v)$$

denote the cdfs of the DM's random utility from choosing actions a or b, respectively.[8] Given these two distributions, we introduce a familiar definition.

Definition 4.3. F_a^x Single-crosses F_b^x from below, $F_a^x \succeq_{sc} F_b^x$, if there exists a $\bar{v} \in \mathbb{R}$ such that for all $v \in \mathbb{R}$, $v \leq \bar{v}$ implies $F_b^x(v) \geq F_a^x(v)$ and $v \geq \bar{v}$ implies $F_a^x(v) \geq F_b^x(v)$.

Then, Hammond III (1974), Lambert and Hey (1979), Karlin and Novikoff (1963), and Jewitt (1987) all contain the following result that highlights the importance of single-crossing in understanding risk aversion.[9] From Jewitt (1987),

[8] Note that the belief x pins down the utility distribution and, therefore, F_a^x and F_b^x.

[9] The literature defines single-crossing in terms of consequences, then posits a strictly increasing utility function. We define the concept in terms of utils, with no loss of generality.

Theorem 4.4. *$F_a^x \succeq_{sc} F_b^x$ implies*

$$\int v dF_a^x(v) \geq \int v dF_b^x(v) \quad \Rightarrow \quad \int \phi(v) dF_a^x(v) \geq \int \phi(v) dF_b^x(v)$$

whenever ϕ is increasing and concave.

To reiterate, single-crossing is a distribution-specific property. It is a statement about cdfs–in the economic context, a characteristic of lotteries. Accordingly, the classical results are comparing specific distributions. In contrast, our concept of safety is prior-free–it is a statement merely about (state-dependent) payoffs. Nevertheless, we can still connect safety to single-crossing. Then,

Lemma 4.5. *If $F_a^x \succeq_{sc} F_b^x$ for all $x \in \Delta$, then $a \succeq_S b$.*

Proof. Suppose for the sake of contraposition that $a \not\succeq_S b$. Consequently, recalling that neither a nor b is weakly dominated, there exists an $x \in \Delta$ and a strictly monotone concave transformation ϕ such that $\mathbb{E}_x u(a,\theta) \geq \mathbb{E}_x u(b,\theta)$, but $\mathbb{E}_x \phi \circ u(a,\theta) < \mathbb{E}_x \phi \circ u(b,\theta)$. Thus, $F_a^x \not\succeq_{sc} F_b^x$. ■

That is, if the entire family of functions $\left\{F_a^x - F_b^x\right\}_{x \in \Delta}$ crosses the horizontal axis at most once (and from below), action a is safer than b. Furthermore, the converse to this lemma is also true.

Lemma 4.6. *If $a \succeq_S b$, then $F_a^x \succeq_{sc} F_b^x$ for all $x \in \Delta$.*

Proof. See Appendix A.4. ■

We see that safety implies a sense of robust single-crossing: every element of the set $\left\{F_a^x - F_b^x\right\}_{x \in \Delta}$ crosses the horizontal axis at most once from below. Why does this lemma hold? Notably, the conditions equivalent to safety are quite demanding: in every pair of states in which different actions are optimal, the payoffs to the safer action lie within the convex hull of the payoffs to the other. This implies that the cdf over utils for the unsafe action must be supported on values below and above the support of the safe action's cdf, *but nothing in between*. Hence, single-crossing. Combining Lemmas 4.5 and 4.6 yields

Theorem 4.7. *$a \succeq_S b$ if and only if $F_a^x \succeq_{sc} F_b^x$ for all $x \in \Delta$.*

4.3 Ordering the Action Set

The "safer than" relation is not, in general, transitive when there are three or more states. This lack of general transitivity indicates that if the set of actions is to be partially or totally ordered by safety, additional conditions are needed.[10]

Proposition 4.8. *Let the set of states, Θ, be totally ordered. If*

(i) For all actions $a \in A$, the DM's utility $u(a, \theta)$ is monotone in θ; and

(ii) For any two actions $a, b \in A$, the states are ordered by optimality, i.e., either $\sup_\theta \mathcal{A} \leq \inf_{\theta'} \mathcal{B}$ or $\inf_\theta \mathcal{A} \geq \sup_{\theta'} \mathcal{B}$,

then the set of actions, A, is totally ordered by $\succeq_S$.

In short, we are putting enough structure on payoffs to allow for a thorough comparison of actions. Recall Corollary 3.3 which says that if payoffs are monotone, then for any two actions a and b, the slope of the payoff of a being smaller is equivalent to a being safer than b. The same rationale applies here, and we are able to rank all actions with respect to slope (more-or-less) and therefore safety.

4.4 Smooth Ambiguity Aversion

Our main result extends in a natural way to the smooth ambiguity model of Klibanoff, Marinacci, and Mukerji (2005). Suppose our DM prefers action a to action b if and only if

$$\mathbb{E}_\nu \psi\left(\int u(a, \theta) d\pi(\theta)\right) \geq \mathbb{E}_\nu \psi\left(\int u(b, \theta) d\pi(\theta)\right),$$

[10] In Appendix B, we present an example with a quadratic loss utility function demonstrating Theorem 4.1 and Proposition 4.8.

where ψ is a monotone concave function and $\nu \in \Delta\Pi$ is a distribution over feasible probability measures $\pi \in \Pi \subseteq \Delta\Theta$. We specify that Π is compact.

Following Klibanoff et al. (2005), our DM becomes more ambiguity averse if the internal von Neumann–Morgenstern utility u stays unchanged and ψ transforms to $\hat{\psi} \equiv \phi \circ \psi$, where ϕ is some monotone concave function.

Understanding $\psi\left(\int u(a,\theta)d\pi(\theta)\right)$ as a concave functional $\psi\colon A \times \Delta\Theta \to \mathbb{R}_+$, we define $\mathcal{A} \subset \Delta\Theta$ to be the set of priors at which a is uniquely optimal and $\mathcal{B} = \Delta\Theta \setminus \mathcal{A}$ to be the set in which b is uniquely optimal. We further define

$$\alpha_\pi := \psi\left(\int u(a,\theta)d\pi(\theta)\right) \quad \text{and} \quad \beta_\pi := \psi\left(\int u(b,\theta)d\pi(\theta)\right);$$

then, applying Theorem 4.1, obtain

Proposition 4.9. *Action a is safer than action b if and only if for each $\pi \in \mathcal{A}$ and $\pi' \in \mathcal{B}$, $\beta_{\pi'} \geq \alpha_\pi$ and $\alpha_{\pi'} \geq \beta_\pi$.*

5 Applications

Our relation is useful in a variety of settings. In §5.1, we formulate a notion of robust hedging, and §5.2 uses our safer-than relation to compare securities. §5.3 reveals how our relation can be used to formulate a notion of safe strategy profiles in games.

5.1 Hedging

Consider the classic question of constructing an asset portfolio to hedge against risk. To be concrete, suppose our DM's wealth, or other holdings, y, fluctuates according to market conditions (the state θ) and is distributed according to H_θ. She decides whether to add asset a or asset b to her portfolio to hedge against the unavoidable variation of y. Asset a pays w_θ in state θ and asset b pays v_θ.

We say that Asset a hedges risk better than asset b if for any strictly monotone concave utility function u, the set of beliefs at which a is preferred by the DM to b is a superset of the set of beliefs at which a is preferred to b by a risk-neutral DM.

We can put this problem in the language of the earlier framework, setting

$$\alpha_\theta = \int (w_\theta + y)\, dH_\theta(y) = w_\theta + \mu_\theta,$$

where $\mu_\theta := \mathbb{E}_{H_\theta}[Y]$, and

$$\beta_\theta = v_\theta + \mu_\theta.$$

Appealing to genericity we stipulate $w_\theta > v_\theta$ for all $\theta \in \mathcal{A}$ and $v_{\theta'} > w_{\theta'}$ for all $\theta' \in \mathcal{B} \equiv \Theta \setminus \mathcal{A}$. Thus,

Proposition 5.1. *If $w_{\theta'} \geq v_\theta$, $v_{\theta'} \geq w_\theta$, and $H_{\theta'}$ first-order stochastically dominates H_θ for all $\theta \in \mathcal{A}$ and $\theta' \in \mathcal{B}$, then asset a hedges risk better than asset b.*

The sufficient condition above translates Theorem 4.1 into the hedging setting. We must still have $\beta_{\theta'} \geq \alpha_\theta$ and $\alpha_{\theta'} \geq \beta_\theta$, but we also need to discipline the conditional distributions of the aggregate risk. The intuition for the stochastic dominance assumptions is as follows. Asset a, by construction, is optimal in all states $\theta \in \mathcal{A}$. If the state is some $\theta' \in \mathcal{B}$, the DM has chosen incorrectly, but because $H_{\theta'}$ dominates H_θ, the *downside risk* from choosing incorrectly is mitigated. In this sense, asset a yields more insurance to the DM, and, therefore, hedges risk better.

The classical literature–e.g., Kihlstrom, Romer, and Williams (1981), Ross (1981), Nachman (1982), Jewitt (1987), and Eeckhoudt, Gollier, and Schlesinger (1996)–studies a similar question, asking when

$$\mathbb{E}_{F_a^x,H} u(Y_a + V) \geq \mathbb{E}_{F_b^x,H} u(Y_b + V) \quad \Rightarrow \quad \mathbb{E}_{F_a^x,H} \phi \circ u(Y_a + V) \geq \mathbb{E}_{F_b^x,H} \phi \circ u(Y_b + V),$$

for any monotone concave ϕ, where random variable V (with cdf H) is the DM's random wealth and Y_a and Y_b (with respective cdfs F_a^x and F_b^x) are the random payoffs from assets a and b, respectively. Our Proposition 5.1 is different from these works in two essential ways: first, the aforementioned papers are concerned with the properties of random variables that lead to one asset being more sensitive to an investor's risk preferences. In contrast, ours is distribution-free comparison between assets. Second, the papers in the literature stipulate that Y_i and V are independent. We assume no such independence here.

5.2 Safe Securities

Consider a firm that is selling state-contingent securities to investors in order to raise capital. In state $\theta \in [0,1]$, the firm receives cash flow θ and pays the promised security $S(\theta)$ to the investor, with the distribution of states denoted $x \in \Delta([0,1])$. These securities can be structured in a variety of ways, and many papers in the literature examine which type of security is best for either the firm to sell or the investor to offer, given the specific context. We use the tools that we have developed to compare the robustness to risk aversion of securities.

While there is an abundance of ways to structure the state-contingent payoff $S(\theta)$, here are three of the most common:

1. Equity: the investor receives a constant portion, $\eta \in (0,1)$, of the cash flow, or $S(\theta) = \eta\theta$,
2. Debt: the investor is owed a debt $d \in (0,1)$ and collects as much of it as possible, or $S(\theta) = \min\{\theta, d\}$; and
3. Call Option: the investor gets a call option with a strike price of $\rho \in (0,1)$, or $S(\theta) = \max\{\theta - \rho, 0\}$.

Following the literature–see, e.g., Nachman and Noe (1994) and DeMarzo, Kremer, and Skrzypacz (2005)– we assume that any security $S(\theta)$ satisfies the following properties:

(i) Monotonicity I: S (the investor's share of cash flow) is nondecreasing in θ;

(ii) Monotonicity II: $\theta - S$ (the firm's share of cash flow) is nondecreasing in θ;

(iii) Limited liability: $0 \leq S(\theta) \leq \theta$.

We scrutinize securities from the perspective of an investor with a strictly increasing utility function over wealth: $u\colon W \to \mathbb{R}$. Consider two securities, S_a and S_b, and assume that neither is weakly dominated: there exist realizations of the random cash flow under which each is strictly preferred (*ex post*) to the other.

Define

$$\mathcal{A} := \{\theta \in [0,1] : u(S_a(\theta)) > u(S_b(\theta))\}, \quad \mathcal{B} := \{\theta' \in [0,1] : u(S_b(\theta')) > u(S_a(\theta'))\}.$$

By Theorem 4.1, security S_a is safer than S_b if and only if for all $\theta \in \mathcal{A}$ and $\theta' \in \mathcal{B}$, $u(S_b(\theta')) \geq u(S_a(\theta))$ and $u(S_a(\theta')) \geq u(S_b(\theta))$. Noting that our definition of single-crossing from §4.2 is now in terms of cash flows rather than utils, we have

Theorem 5.2. *$S_a \geq_S S_b$ if and only if $S_b \geq_{sc} S_a$.*

Proof. ($\Rightarrow$) Let $S_b \geq_{sc} S_a$. Then for all $\theta \in \mathcal{A}$ and $\theta' \in \mathcal{B}$, $u(S_b(\theta')) > u(S_a(\theta')) \geq u(S_a(\theta))$ and $u(S_a(\theta')) \geq u(S_a(\theta)) > u(S_b(\theta))$, where we used S_a's monotonicity.

($\Leftarrow$) Suppose for the sake of contraposition $S_b \not\geq_{sc} S_a$. This means there exist $\theta_1 < \theta_2$ such that $S_a(\theta_1) < S_b(\theta_1)$ and $S_b(\theta_2) < S_a(\theta_2)$. By the monotonicity of S_b, $S_b(\theta_1) \leq S_b(\theta_2) < S_a(\theta_2)$, so $S_a \not\geq_S S_b$. ■

Debt yields the highest possible payoff for low states ($S(\theta) = \theta$), and as such, debt must single-cross any other security from above. Thus,

Corollary 5.3. *Debt is safer than any other security.*

Intuitively, a security is safer if it gives a constant payoff, regardless of the state. This is constrained, however, by limited liability ($S(\theta) \leq \theta$). A call option yields the lowest possible payoff for low states ($S(\theta) = 0$), and has the largest possible slope when it is non-constant. Consequently, by the intermediate value theorem, a call option must single-cross from below any other security. So,

Corollary 5.4. *All securities are safer than a call option.*

As observed by, for instance, DeMarzo et al. (2005), Dang, Gorton, and Holmström (2013), and Inostroza and Tsoy (2022), a security that crosses the other from below is more information sensitive: the value of information about the state is higher. We show that this precise means of comparison (signal-crossing from below), is the way to rank securities in terms of their robustness to risk aversion.

5.3 A Different Kind of Risk Dominance in Games

As games are just decision problems with endogenous payoffs, our results may be applied to strategic settings. Weinstein (2016) reveals that increased risk aversion expands the set of rationalizable strategies. Here, we show that although all strategies remain rationalizable, some become more rationalizable than others in the sense that they can be rationalized by more beliefs than before.

Consider the following two-player, two-action coordination game. If both players choose action a, they each get a payoff of α_1; and if both players choose action b, they each get a payoff of β_2. Their mismatch payoffs are $u_1(a,b) = u_2(b,a) = \alpha_2$ and $u_1(b,a) = u_2(a,b) = \beta_1$. We assume that $\alpha_1 > \beta_1$ and $\beta_2 > \alpha_2$.

We say that a strategy pair (a_1, a_2) is Safe if the set of beliefs $(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2$ with respect to which a_1 and a_2 are best responses increases in size (in the set-inclusion sense) as players become more risk-averse. Then, from Proposition 3.1,

Proposition 5.5. $\alpha_2 \geq \beta_1$ *and* $\beta_2 \geq \alpha_1$ *if and only if* (a,a) *is safe.*

The necessary and sufficient condition in this proposition differs from the risk-dominance condition of Harsanyi and Selten (1988); which reduces to (a,a) risk dominates (b,b) if $\beta_2 - \alpha_2 \leq \alpha_1 - \beta_1$. Our concept of safety, therefore, requires a weaker condition relative to risk dominance.

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A Omitted Proofs

A.1 Proposition 3.1 Proof

Proof. ($\Leftarrow$) Now suppose for the sake of contraposition that $\beta_0 > \alpha_1$ (and recall $\alpha_0 > \beta_0$). There are two possibilities: either $\alpha_0 \leq \beta_1$, or $\alpha_0 > \beta_1$.

Suppose first $\alpha_0 \leq \beta_1$, so

$$\beta_1 \geq \alpha_0 > \beta_0 > \alpha_1.$$

Let

$$\phi(y) = \min\{y, ky + c\},$$

where

$$c = \frac{\beta_0(\beta_0\beta_1 - \alpha_0\alpha_1)}{\beta_0(\alpha_1 - \beta_0) + \alpha_0(\beta_0 - 2\alpha_1) + \beta_0\beta_1} \text{ and } k = \frac{(\alpha_0 - \beta_0)(\beta_0 - \alpha_1)}{\beta_0(\alpha_1 - \beta_0) + \alpha_0(\beta_0 - 2\alpha_1) + \beta_0\beta_1}.$$

It is straightforward to check that $k \in (0,1)$ and $k\beta_0 + c = \beta_0$, so ϕ is weakly concave, as required.

Moreover,

$$\frac{\phi(\beta_1) - \phi(\alpha_1)}{\beta_1 - \alpha_1} > \frac{\phi(\alpha_0) - \phi(\beta_0)}{\alpha_0 - \beta_0} \Leftrightarrow \frac{k\beta_1 + c - \alpha_1}{\beta_1 - \alpha_1} - \frac{k\alpha_0 + c - \beta_0}{\alpha_0 - \beta_0} > 0,$$

if and only if

$$(1 - k)(\beta_0\beta_1 - \alpha_0\alpha_1) - c(\beta_1 - \alpha_0 + \beta_0 - \alpha_1) > 0,$$

which also holds.

Finally, suppose $\alpha_0 > \beta_1$, in which case we have $\alpha_0 > \beta_1 > \alpha_1$ and $\alpha_0 > \beta_0 > \alpha_1$.

By the three-chord lemma, we have

$$\frac{\phi(\beta_1)-\phi(\alpha_1)}{\beta_1-\alpha_1} \geq \frac{\phi(\alpha_0)-\phi(\alpha_1)}{\alpha_0-\alpha_1},$$

so it suffices to construct a concave monotone ϕ for which

$$\Psi := \frac{\phi(\alpha_0)-\phi(\alpha_1)}{\alpha_0-\alpha_1} - \frac{\phi(\alpha_0)-\phi(\beta_0)}{\alpha_0-\beta_0} > 0.$$

To that end, let

$$\phi(y) = \min\left\{y, \frac{y+\beta_0}{2}\right\}.$$

Plugging this in, we have

$$\Psi = \frac{\frac{\alpha_0+\beta_0}{2}-\alpha_1}{\alpha_0-\alpha_1} - \frac{\frac{\alpha_0+\beta_0}{2}-\beta_0}{\alpha_0-\beta_0} = \frac{\beta_0-\alpha_1}{2(\alpha_0-\alpha_1)} > 0,$$

as desired. ■

A.2 Corollary 3.2 Proof

Proof. ($\not\Leftarrow$) The following example suffices: $\alpha_0 = 5$, $\beta_1 = 4$, $\alpha_1 = 3$, and $\beta_0 = 1$. Then, $\alpha_0 > \beta_1$ so $a \not\succeq_S b$ but

$$|\alpha_1-\alpha_0| = 2 \leq 3 = |\beta_1-\beta_0|.$$

($\Rightarrow$) If $a \succeq_S b$, Proposition 3.1 implies $\beta_1 \geq \alpha_0$ and $\beta_1 > \alpha_1 \geq \beta_0$. Consequently,

$$|\beta_1-\beta_0| = \beta_1-\beta_0 > \alpha_1-\alpha_0,$$

so if $\alpha_0 \leq \alpha_1$ we are done. Now let $\alpha_0 > \alpha_1$ and suppose for the sake of contradiction

$$\alpha_0-\alpha_1 > \beta_1-\beta_0,$$

which holds if and only if

$$\alpha_0 + \beta_0 > \alpha_1 + \beta_1,$$

which is false. ■

A.3 Theorem 4.1 Proof

Proof. We start by constructing a number of objects generated when the DM's utility is u, understanding their analogs for utility $\hat{u} = \phi \circ u$ to be generated in the same manner.

Let $H_{a,b}$ denote the hyperplane of indifference, the set of beliefs at which the DM is indifferent between a and b:

$$H_{a,b} := \{x \in \Delta \colon \mathbb{E}_x u(a,\theta) = \mathbb{E}_x u(b,\theta)\}.$$

By our genericity assumption, this hyperplane does not intersect the boundary of the simplex at a vertex.

For any $\theta \in \Theta$, let v_θ denote the corresponding vertex (as a point in the simplex). Furthermore, for any $\theta \in \mathcal{A}$ and $\theta' \in \mathcal{B}$, let $e_{\theta,\theta'}$ denote the edge of Δ "between the two states:"

$$e_{\theta,\theta'} := \{x \in \Delta | \exists \lambda \in (0,1) \colon \lambda v_\theta + (1-\lambda) v_{\theta'} = x\},$$

and let $x_{\theta,\theta'}$ denote the point of indifference between the two actions lying on the edge $e_{\theta,\theta'}$:

$$x_{\theta,\theta'} := \left\{x \in e_{\theta,\theta'} \colon \mathbb{E}_x u(a,\theta) = \mathbb{E}_x u(b,\theta)\right\}.$$

Equivalently, $x_{\theta,\theta'} = e_{\theta,\theta'} \cap H_{a,b}$.

$P_{a,b}(a)$ is a convex set, so by the Krein-Milman theorem, it is the closed convex hull of its extreme points. Furthermore, its set of extreme points is the set

$$\underline{\mathrm{P}} := \{v_\theta \colon \theta \in \mathcal{A}\} \cup \{x_{\theta,\theta'} \colon \theta \in \mathcal{A},\ \theta' \in \mathcal{B}\},$$

i.e., the set of vertices at which a is uniquely optimal, and the points on the edges connecting the vertices at which a is uniquely optimal with the vertices at which b is uniquely optimal.

Now, $P_{a,b}(a) \subseteq \hat{P}_{a,b}(b)$ if and only if $\underline{\mathrm{P}} \subseteq \mathrm{cch}\,\hat{\underline{\mathrm{P}}}$, i.e., the set $\underline{\mathrm{P}}$ lies within the closed convex hull of the set $\hat{\underline{\mathrm{P}}}$. Moreover, by construction $\underline{\mathrm{P}} \subseteq \mathrm{cch}\,\hat{\underline{\mathrm{P}}}$ if and only if for each $\theta \in \mathcal{A}$ and $\theta' \in \mathcal{B}$, there exists some $\lambda \in (0,1]$ such that $x_{\theta,\theta'} = \lambda \hat{x}_{\theta,\theta'} + (1-\lambda) v_\theta$. That is, for each edge containing a point of indifference, the indifference point for utility u, $x_{\theta,\theta'}$, must lie between the indifference point in decision problem for utility $\hat{u}$ and the vertex v_θ.

By Proposition 3.1, this holds for any strictly monotone concave transformation of u, ϕ, if and only if for all $\theta \in \mathcal{A}$ and $\theta' \in \mathcal{B}$, $\beta_{\theta'} \geq \alpha_\theta$ and $\alpha_{\theta'} \geq \beta_\theta$. ■

A.4 Lemma 4.6 Proof

Proof. Observe that if x is such that F_a^x and F_b^x can be ranked according to first-order stochastic dominance, single-crossing holds trivially. Now let us assume x is such that F_a^x and F_b^x are FOSD-incomparable.

We define the following sets:

$$\mathcal{A}^\circ := \{\theta \in \mathcal{A} \colon \beta_\theta < \alpha_{\theta'}\ \forall\ \theta' \in \mathcal{B}\}, \quad \mathcal{A}^\dagger := \{\theta \in \mathcal{A} \colon \exists \theta' \in \mathcal{B} \colon \beta_\theta = \alpha_{\theta'}\},$$

$$\mathcal{B}^\dagger := \{\theta' \in \mathcal{B} \colon \exists \theta \in \mathcal{A} \colon \beta_\theta = \alpha_{\theta'}\}, \quad \mathcal{B}^\triangle := \{\theta' \in \mathcal{B} \colon \beta_\theta < \alpha_{\theta'}\ \forall\ \theta \in \mathcal{A}\},$$

$$\mathcal{A}^\triangle := \{\theta \in \mathcal{A} \colon \beta_{\theta'} > \alpha_\theta\ \forall\ \theta' \in \mathcal{B}\}, \quad \mathcal{A}^\ddagger := \{\theta \in \mathcal{A} \colon \exists \theta' \in \mathcal{B} \colon \beta_{\theta'} = \alpha_\theta\},$$

$$\mathcal{B}^\ddagger := \{\theta' \in \mathcal{B} \colon \exists \theta \in \mathcal{A} \colon \beta_{\theta'} = \alpha_\theta\}, \ \&\ \mathcal{B}^\circ := \{\theta' \in \mathcal{B} \colon \beta_{\theta'} > \alpha_\theta\ \forall\ \theta \in \mathcal{A}\}.$$

We assume without loss of generality that each set is nonempty.

Given some $x \in \Delta$ that is such that denominators of the following fractions are strictly positive (which is WLOG), we define

$$b_0 := \frac{\int_{\theta \in \mathcal{A}^\circ} \beta_\theta dx(\theta)}{\int_{\theta \in \mathcal{A}^\circ} dx(\theta)}, \quad b_1 := \frac{\int_{\theta \in \mathcal{A}^\dagger} \beta_\theta dx(\theta)}{\int_{\theta \in \mathcal{A}^\dagger} dx(\theta)}, \quad b_2 := \frac{\int_{\theta' \in \mathcal{B}^\ddagger} \beta_{\theta'} dx(\theta')}{\int_{\theta' \in \mathcal{B}^\ddagger} dx(\theta')},$$

$$b_3 := \frac{\int_{\theta' \in \mathcal{B}^\circ} \beta_{\theta'} dx(\theta')}{\int_{\theta' \in \mathcal{B}^\circ} dx(\theta')},$$

$$a_0 := \frac{\int_{\theta' \in \mathcal{B}^\dagger} \alpha_{\theta'} dx(\theta')}{\int_{\theta' \in \mathcal{B}^\dagger} dx(\theta')}, \quad a_1 := \frac{\int_{\theta' \in \mathcal{B}^\triangle} \alpha_{\theta'} dx(\theta')}{\int_{\theta' \in \mathcal{B}^\triangle} dx(\theta')}, \quad a_2 := \frac{\int_{\theta \in \mathcal{A}^\triangle} \alpha_\theta dx(\theta)}{\int_{\theta \in \mathcal{A}^\triangle} dx(\theta)},$$

and

$$a_3 := \frac{\int_{\theta \in \mathcal{A}^\ddagger} \alpha_\theta dx(\theta)}{\int_{\theta \in \mathcal{A}^\ddagger} dx(\theta)}.$$

Note that by construction

$$b_0 < b_1 = a_0 < a_1, a_2 < a_3 = b_2 < b_3.$$

Evidently, $F_a^x(u) \le F_b^x(u)$ for all $u < b_1$. Then, either i. $F_a^x(b_1) > F_b^x(b_1)$ or ii. $F_a^x(b_1) \le F_b^x(b_1)$. In the first case, we have $F_a^x(u) > F_b^x(u)$ for all $b_1 \le u < b_2$. Moreover, $1 = F_a^x(u) \ge F_b^x(u)$ for all $u \ge b_2$. Single-crossing (from below) is thus confirmed. In the second case, we still have $1 = F_a^x(u) \ge F_b^x(u)$ for all $u \ge b_2$, so F_a^x crosses F_b^x once from below at some $u \in (b_1, b_2]$. ∎

A.5 Proposition 4.8 Proof

Proof. Recall

$$\mathcal{A} \equiv \{\theta \in \Theta \colon \alpha_\theta > \beta_\theta\} \quad \text{and} \quad \mathcal{B} \equiv \Theta \setminus \mathcal{A} = \{\theta' \in \Theta \colon \beta_{\theta'} > \alpha_{\theta'}\};$$

similarly, we define

$$\tilde{\mathcal{B}} := \left\{\tilde{\theta} \in \Theta : \beta_{\tilde{\theta}} > \tau_{\tilde{\theta}}\right\} \quad \text{and} \quad \tilde{\mathcal{C}} \equiv \Theta \setminus \tilde{\mathcal{B}} = \left\{\tilde{\theta}' \in \Theta : \tau_{\tilde{\theta}'} > \beta_{\tilde{\theta}'}\right\}.$$

Strongly Connectedness: Take an arbitrary a and b, where WLOG $\sup_\theta \mathcal{A} \le \inf_{\theta'} \mathcal{B}$. Then, for all $\theta \in \mathcal{A}$, $\theta' \in \mathcal{B}$,

$$\beta_\theta < \alpha_\theta \overset{\text{Monotonicity of } u}{\le} \alpha_{\theta'} < \beta_{\theta'},$$

so $a \succeq_S b$.

Transitivity: Suppose $a \succeq_S b$ and $b \succeq_S c$, which imply $\sup_\theta \mathcal{A} \le \inf_{\theta'} \mathcal{B}$ and $\sup_{\tilde{\theta}} \mathcal{B} \le \inf_{\tilde{\theta}'} \mathcal{C}$. Consequently, for all $\tilde{\theta} \in \tilde{\mathcal{B}}$ and $\tilde{\theta}' \in \tilde{\mathcal{C}}$,

$$\tau_{\tilde{\theta}} < \beta_{\tilde{\theta}} \le \beta_{\tilde{\theta}'} < \tau_{\tilde{\theta}'}.$$

This immediately implies for all $\theta \in \mathcal{A}$, $\theta' \in \mathcal{B}$, $\tilde{\theta} \in \tilde{B}$, and $\tilde{\theta}' \in \tilde{\mathcal{C}}$,

$$\tau_{\tilde{\theta}} \overset{b \succeq_S c}{\le} \beta_\theta < \alpha_\theta \overset{\text{Monotonicity of } u}{\le} \alpha_{\theta'} < \beta_{\theta'} \overset{b \succeq_S c}{\le} \tau_{\tilde{\theta}'},$$

i.e., that $a \succeq_S c$.

Antisymmetry and reflexivity of $\succeq_S$ are immediate. ■

A.6 Proposition 5.1 Proof

Proof. First,

Claim A.1. $\int [u(w_\theta + y) - u(v_\theta + y)]\, dH_\theta(y) \ge \int [u(w_\theta + y) - u(v_\theta + y)]\, dH_{\theta'}(y)$.

Proof. Naturally, this is equivalent to

$$\int [u(v_\theta + y) - u(w_\theta + y)]\, dH_{\theta'}(y) \ge \int [u(v_\theta + y) - u(w_\theta + y)]\, dH_\theta(y).$$

Observe that for any y,

$$\frac{d}{dy}\left[u\left(v_\theta+y\right)-u\left(w_\theta+y\right)\right]=u'\left(v_\theta+y\right)-u'\left(w_\theta+y\right)\geq 0,$$

by the concavity of u plus the fact that $w_\theta > v_\theta$. ■

Second,

Claim A.2. *$\phi(z) := \int u(z+y)\, dH_1(y)$ is a concave function of z.*

Proof. Directly

$$\begin{aligned}\phi\left(\lambda z_1+(1-\lambda)z_2\right)&=\int\left[u\left(\lambda z_1+(1-\lambda)z_2+y\right)\right]dH_1(y)\\&\geq\int\left[\lambda u\left(z_1+y\right)+(1-\lambda)u\left(z_2+y\right)\right]dH_1(y),\end{aligned}$$

by the concavity of u plus the fact that everything is positive. ■

Finally, Proposition 3.1 plus Claims A.1 and A.2, yield

$$\begin{aligned}\frac{\int u\left(w_\theta+y\right)dH_\theta(y)-\int u\left(v_\theta+y\right)dH_\theta(y)}{w_\theta-v_\theta}&\geq\frac{\phi\left(w_\theta\right)-\phi\left(v_\theta\right)}{w_\theta-v_\theta}\\&\geq\frac{\phi\left(v_{\theta'}\right)-\phi\left(w_{\theta'}\right)}{v_{\theta'}-w_{\theta'}}\\&=\frac{\int u\left(v_{\theta'}+y\right)dH_{\theta'}(y)-\int u\left(w_{\theta'}+y\right)dH_{\theta'}(y)}{v_{\theta'}-w_{\theta'}},\end{aligned}$$

as desired. ■

B Quadratic Loss Examples

To make the concepts concrete, we consider a quadratic loss utility function, first with two states, then with a continuum of states.

Let $\theta \in \{0,1\}$, $A=[0,1]$, and $u(a,\theta)=1-(a-\theta)^2$. We will first characterize the set of actions that are safer than an arbitrary action $a \leq \frac{1}{2}$. Observe that $u(a,0) > u(a,1)$

if $a < \frac{1}{2}$, so that for any two actions $b < a \leq \frac{1}{2}$, expected utility is monotone, and $|\gamma_a| < |\gamma_b|$ so that Proposition 3.3 implies that a is safer than b. Next, let $a \leq \frac{1}{2} < b$, so that $u(a,0) \leq u(b,1)$ and $u(b,0) \leq u(a,1)$ iff $a \geq 1-b$. Accordingly, for an $a \leq \frac{1}{2}$, a is safer than b iff $a \geq b$ or $a \geq 1-b$. In the monotone pieces of V, clearly actions are totally ordered, but because V is not monotone overall, we cannot rank all actions in terms of safety. Rather, it depends on how far away each is from the “safest" action $\frac{1}{2}$.

Now consider the case where $\Theta = A = [0,1]$. We demonstrate the mechanics of Proposition 4.8 by comparing a utility function that does not satisfy the conditions of the proposition to one that does. To begin, let $u(a,\theta) = 1-(a-\theta)^2$.

Remark B.1. With utility $u(a,\theta) = 1-(a-\theta)^2$, no two distinct actions can be ranked according to $\succeq_S$.

Proof. Let $a < b$. Observe that the state in which the DM is indifferent between the two actions is $\hat{\theta} = \frac{a+b}{2}$, so that $\mathcal{A} = \left[0, \frac{a+b}{2}\right]$ and $\mathcal{B} = \left[\frac{a+b}{2}, 1\right]$. Now let us compare an arbitrary $\theta \leq \hat{\theta}$ with $\theta' \geq \hat{\theta}$. Observe that

$$\alpha_\theta = 1-(a-\theta)^2 \geq \beta_{\theta'} = 1-(b-\theta')^2 \iff$$
$$\left|b-\theta'\right| \geq |a-\theta|,$$

but neither direction of this inequality holds for all $\theta \in \mathcal{A}$ and $\theta' \in \mathcal{B}$. ■

However, a slight tweak to the decision problem, one that leaves the DM's behavior unaltered, satisfies the assumptions of Proposition 4.8, and therefore $\succeq_S$ totally orders the actions. Now let

$$u(a,\theta) = 1-(a-\theta)^2+\theta^2,$$

and consider an arbitrary pair of actions $a < b$. As before, the state in which the DM is indifferent between the two actions is $\hat{\theta} = \frac{a+b}{2}$, and $\mathcal{A}$ and $\mathcal{B}$ are unchanged.

To order a and b according to $\geq_S$, we need for every $\theta \leq \hat{\theta}$ and $\theta' \geq \hat{\theta}$ to satisfy

$$\alpha_\theta = 1 - a^2 + 2a\theta \leq \beta_{\theta'} = 1 - b^2 + 2b\theta' \quad \Leftrightarrow \quad \theta' \geq \frac{b^2 - a^2}{2b} + \frac{a}{b}\theta,$$

and

$$\alpha_{\theta'} = 1 - a^2 + 2a\theta' \geq \beta_\theta = 1 - b^2 + 2b\theta \quad \Leftrightarrow \quad \theta' \geq \frac{a^2 - b^2}{2a} + \frac{b}{a}\theta,$$

which always hold. Consequently, $a \geq_S b$ if and only if $a \leq b$. As $a \leq b$ and $b \leq a$ if and only if $a = b$, $\geq_S$ is reflexive. Furthermore, $\geq$, and so $\geq_S$, are strongly connected and transitive, so $\geq_S$ is a total order over actions.