Extreme & Exposed Points of Fusions and Their Economic Applications

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January 8, 2024

Motivation

Kleiner, Moldovanu, & Strack (2022): extreme (and exposed) points of the set of monotonic functions that either majorize, or are majorized by a given, real-valued, monotonic function f defined on an interval.

If the monotonic functions are cdfs, majorization $\equiv$ SOSD.

Many applications: equivalence and optimality of mechanisms for auctions and contests, Bayesian persuasion, delegation, & more.

What about **multi-dimensional** frameworks?

Reminder (figure from Goh et al. 2018)

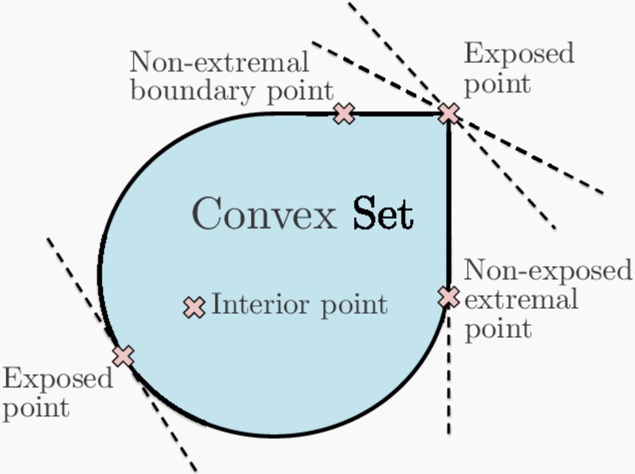


Figure: Extreme & Exposed Points of a Convex Set

Multi-dimensional fusions

Let $X \subseteq \mathbb{R}^d$ be a closed polygon (so convex, compact) and μ an absolutely continuous measure on X (w.r.t. to the Lebesgue measure).

Denote by $\mathcal{F}_\mu$ the set of of mean preserving contractions (or fusions) of μ:

$$\upsilon \in \mathcal{F}_\mu \quad \Leftrightarrow \quad \int f d\upsilon \leq \int f d\mu$$

for all real valued, convex functions f on X.

Probability measures obtained by "mixing" together parts of the measure μ.

Set $\mathcal{F}_\mu$ is compact and convex.

Moment persuasion

Special class of multi-dimensional Bayesian persuasion problems: receiver's (sender-preferred) optimal action and the sender's expected payoff from any such action depend only on the receiver's vector of posterior means.

u is sender's (continuous) reduced-form payoff.

Given prior μ, problem reduces to

$$\max_{\nu \in \mathcal{F}_\mu} \int u(x) d\nu.$$

Linear objective $\Rightarrow$ optimum at an extreme point of $\mathcal{F}_\mu$.

For tractability, focus on extreme and exposed points of $\mathcal{F}_\mu$ that are finitely supported.

Extreme points (necessity)

Definition. Vectors $x_1, x_2, \ldots, x_r \in \mathbb{R}^d$ are Affinely Independent if $x_2 - x_1, x_3 - x_1, \ldots x_r - x_1$ are linearly independent. A set of affinely independent vectors contains at most $d+1$ elements.

Theorem. Let $\upsilon \in \mathcal{F}_\mu$ be a finitely supported extreme point. Then, there exists a partition of X into convex sets $X_1, X_2, \ldots X_K$ such that, for each i, $\upsilon|_{X_i}$ is a fusion of $\mu|_{X_i}$ and $\upsilon|_{X_i}$ has affinely independent support.

(Handwavy) Reminder of 1d results

Kleiner, Moldovanu, & Strack (2021) and Arieli, Babichenko, Smorodinsky, & Yamashita (2023).

Extreme ≡ exposed points for MPCs of 1d μ:

Partition μ's support ($[0,1]$) into a collection of intervals $[\underline{x}_i, \bar{x}_i)$. On each interval, three possibilities:

1. Full revelation: MPC $\upsilon = \mu$ on $[\underline{x}_i, \bar{x}_i)$;
2. No revelation: μ collapsed to its barycenter on $[\underline{x}_i, \bar{x}_i)$; or
3. Binary support: υ has binary support on $[\underline{x}_i, \bar{x}_i)$ w/ $\upsilon|_{[\underline{x}_i, \bar{x}_i)}$ an MPC of $\mu|_{[\underline{x}_i, \bar{x}_i)}$

Why "Bipooling?" **These are the affinely-independent vectors in $\mathbb{R}$!**

Proof sketch

Theorem. Let $\nu \in \mathcal{F}_\mu$ be a finitely supported extreme point. Then, there exists a partition of X into convex sets $X_1, X_2, \ldots X_K$ such that, for each i, $\nu|_{X_i}$ is a fusion of $\mu|_{X_i}$ and $\nu|_{X_i}$ has affinely independent support.

proof sketch. Consider the finest partition such that each element X_i is convex, has positive measure and such that $\nu|_{X_i}$ is a fusion of $\mu|_{X_i}$ for all i.

Show that support of $\nu|_{X_i}$ must be affinely independent. Note that the extreme points of the set of measures $\nu|_{X_i}$ with barycenter b has affinely independent support.

Solution to a different kind of problem

Definition. A fusion ν of μ is Convex Partitional if there exists a partition of X into convex sets $X_1, X_2, \ldots X_K$ such that, for each i, the support of $\nu|_{X_i}$ is a singleton and $\nu|_{X_i}$ is a fusion of $\mu|_{X_i}$.

An extreme point of $\mathcal{F}_\mu$ need not be convex partitional.

However, convex partitional fusions are special...

Maximally informative with a fixed # of points.

Proposition. Suppose $\nu \in \mathcal{F}_\mu$ with K points in its support. Then, there is a convex partitional measure λ with at most K points in its support that satisfies $\nu \preceq \lambda \preceq \mu$; and if ν is not convex partitional, $\nu \prec \lambda$. Moreover, λ can be chosen so that the distribution of the measures of the individual support points is the same as for ν.

proof sketch. G: set of probability measures that are fusions of μ, have at most K points in their support, and have the same distribution of the measures of the individual support points as ν.

Zorn's lemma $\Rightarrow$ There is maximal measure in G according to the convex order.

Perturbation argument $\Rightarrow$ yields convex partitional.

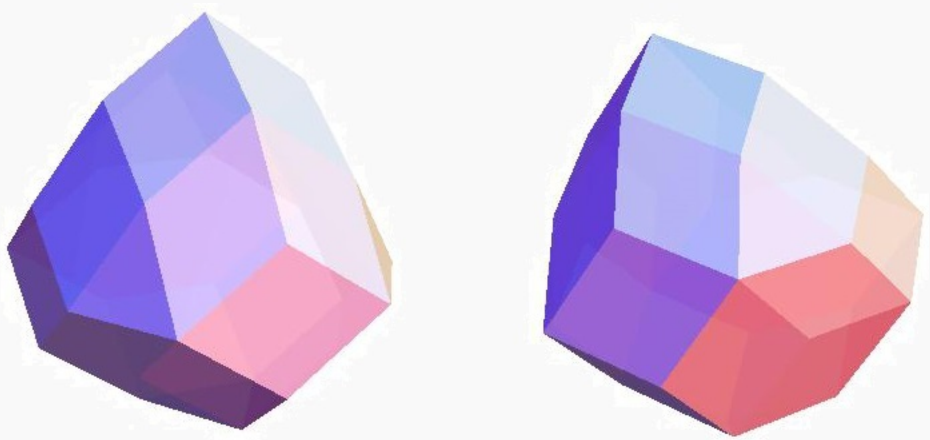
A geometric diversion

“To comport oneself with perfect propriety in Polygonal society, one ought to be a Polygon oneself."

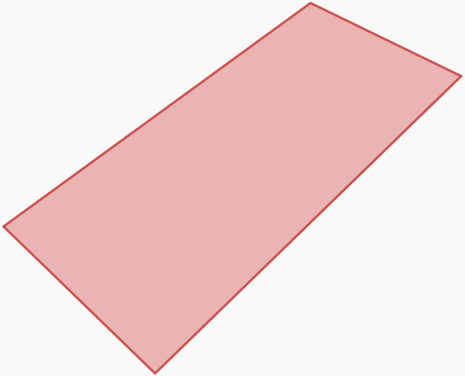
Definition. A bounded subset $P \subseteq \mathbb{R}^d$ is a Polytope if it is the convex hull of a finite set of points $V = \{v_1, \ldots, v_n\}$:

$$P = chV := \left\{ \sum_{i=1}^{n} \lambda_i v_i : \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0, i = 1, \cdots, n \right\}.$$

Polytope



Polytope



Subdivisions

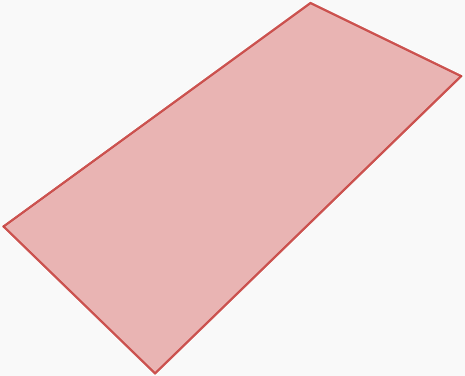
Definition. A Polytopal Complex C in $\mathbb{R}^d$ is a collection of polytopes s.t.:

1. The empty set is in C.
2. For any P in C all faces of P are in C.
3. The intersection of any two polytopes in C is a face of both.

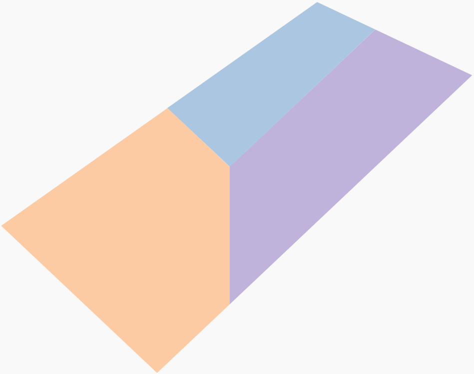
Definition. The union of all polytopes in complex C is called the Underlying Set of C.

Definition. Let P be a polytope. A Polyhedral Subdivision of P is a polytopal complex C with underlying set P.

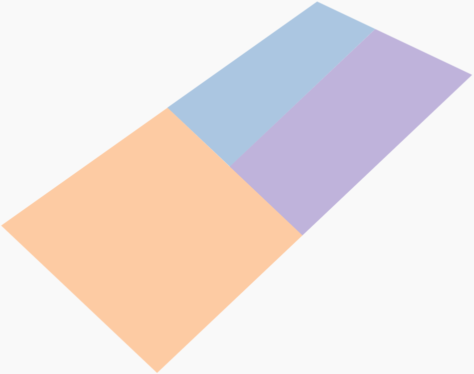
Polytope



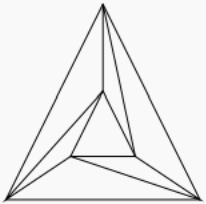
A subdivision



Not a subdivision



A special subdivision



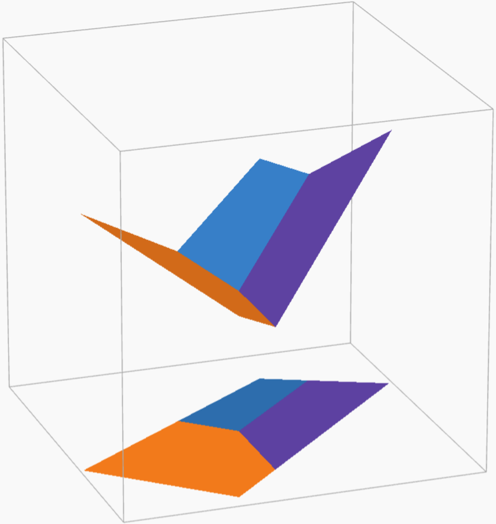
Regular subdivisions

Definition. Let P be a polytope. A Regular Polyhedral Subdivision is a projection on P of a convex polyhedral surface in $\mathbb{R}^{d+1}$.

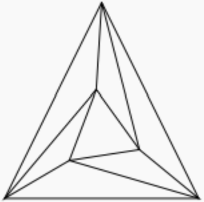
Alternatively,

Definition. A polyhedral subdivision C of V is Regular if there are heights α_i for each point v_i of V such that C is combinatorially isomorphic to the complex of *lower faces* of the polytope $ch\{(v_i, \alpha_i) \in \mathbb{R}^{d+1}, v_i \in V\}$. Here lower faces are in directions with a negative $(d+1)$-th coordinate.

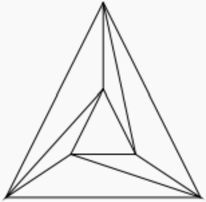
Regular



Which is regular?



(a)



(b)

Exposed points (necessity)

Proposition. Let $\nu \in \mathcal{F}_\mu$ be a finitely supported exposed point. Assume that ν is the unique solution to

$$\max_{\lambda \in \mathcal{F}_\mu} \int u(x) d\lambda(x), \qquad (\star)$$

where u is Lipschitz continuous.

Then there is a regular polyhedral subdivision $\{X_1, \ldots, X_K\}$ of X such that $\mu|_{X_i} \geq \nu|_{X_i}$ for all i and each $\nu|_{X_i}$ has convexly-independent support.

proof sketch. ν is an extreme point $\Rightarrow$ affinely-independent support on partition. Dworczak & Kolotilin (2019) $\Rightarrow$ strong duality holds $\Rightarrow$ existence of piecewise affine and convex function $p \geq u$. So there is a corresponding subdivision.

Exposed (& extreme) points (sufficency)

Proposition. Let $\upsilon \in \mathcal{F}_\mu$ and assume there is a regular polyhedral subdivision $\{X_1, \ldots, X_K\}$ of X such that $\mu|_{X_i} \geq \upsilon|_{X_i}$ for all i and each $\upsilon|_{X_i}$ has affinely-independent support. Then υ is an exposed point of $\mathcal{F}_\mu$.

proof sketch. Recall: polyhedral subdivisions of polytopes in $\mathbb{R}^d \equiv$ projections "down" of convex polyhedral surfaces that live in R^{d+1}.

$\Rightarrow$ There is a convex, piece-wise affine function $p \colon X \to \mathbb{R}$ that projects onto $X_1, X_2, \ldots, X_K$.

Define $u(x) := p(x) - \inf_{y \in \operatorname{supp} \upsilon} ||x - y||$.

By construction, υ is the unique sol to $\max_{\lambda \in \mathcal{F}_\mu} \int u d\lambda$.

Categorization

We categorize things: “red,” “blue,” “orange” describe *families* of colors. Likewise “tall,” and “short.”

In cognitive science, formalizations of this.

Gärdenfors (2004): divide state space into a collection of convex sets, each containing a *prototype* (“red”).

This structure (convexity, single prototype) is *assumed*.

We can provide two micro-foundations for this!

Categorization Justification I

State of the world $X = [0,1]^d$ and μ the prior (admitting a density).

Decision-maker (DM) with finitely many undominated actions. (Almost) WLOG utilities depend on posterior mean. Why?!?

Produces reduced-form value function $V\colon [0,1]^d \to \mathbb{R}$, convex & piecewise affine.

DM acquires info prior to choosing action: (Almost) WLOG mean-measurable cost function $c\colon [0,1]^d \to \mathbb{R}$.

Categorization Justification I

DM solves

$$\max_{\nu \in F_\mu} \int (V - \kappa c) d\nu \quad (\kappa \in \mathbb{R}_{++}).$$

Remark. There exists a $\bar{\kappa} > 0$ such that if the cost parameter $\kappa \leq \bar{\kappa}$, the DM's optimal information acquisition corresponds to categorization with a single prototype per category.

Categorization Justification II

Now DM has K "bins" in which she can place points in the state space.

Memory with finite capacity.

We do not assume structure beyond this: could just assign points in $[0, 1]^d$ to each bin uniformly at random, e.g. Nevertheless...

Remark. There is an optimal categorization that is convex partitional. If the number of undominated actions in the decision problem is weakly greater than the number of possible categories, any optimal categorization must be convex partitional.

Also, think of aligned cheap talk with constraints on message #.