Persuading a Consumer to Visit

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Abstract

We consider a variation on the classic Weitzman search problem: competing firms with products of unknown quality may design how much information a consumer's visit will glean. After observing these information structures, the consumer then decides how to search sequentially across the firms for a high value product. If there are no search frictions, then there is a unique symmetric equilibrium in pure strategies, and the firms are not fully informative. With search frictions, the information a visit will reveal depends in a systematic way on the *ex-ante* probability that a firm's product is high quality. When the expected quality of the product is sufficiently high, there is a unique symmetric equilibrium in which firms are fully informative. There, a small search cost leads to the perfect competition level of information provision–consumers gain when firms are forced to compete on information. Conversely, in the low and medium expected quality cases there are no pure strategy equilibria. Instead, firms mix over a continuum of levels of information: in the low expected quality case they provide full information with probability zero; and in the medium expected quality case they provide full information

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with positive probability. In both cases there are mixed strategy equilibria in which the firms' realized information structures are Blackwell comparable. Moreover, though recall is permitted, in each case there are equilibria in which the consumer never returns.

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1 Introduction

One of the primary ways through which firms attract consumers is by providing information about their products. Of the venues for such information transmission, perhaps the oldest is the physical location of the firm itself, where the firm can allow a prospective buyer to prod, weigh, feel, inspect, and otherwise assess a product. Naturally, technology has expanded the ways such product information can be transmitted; and for some goods, this information can now be transmitted entirely virtually. However, there are many goods that still demand an in-person visit in order for crucial information about the product to be provided.

Products whose appeal to the senses is paramount are primary examples of goods about which essential information cannot be conveyed remotely. The feel of a rug or a car, the fit of clothes or shoes, the sound of an instrument or speakers, the smell of perfume or wooden furniture, or the taste of a food or drink product–these are all things essential to their products that must be observed by the consumer, herself, in the flesh. Moreover, for some items firms are quite forthcoming and provide a great deal of information to consumers who visit. The fine piano purveyor, Steinway & Sons, has practice rooms in its showrooms so that people can get a feel for the instrument themselves, many car dealerships allow test-drives and some even allow prospective buyers to keep the car overnight, upscale clothing stores include changing rooms (and mirrors) for trying on their products, and anyone who has mistakenly wandered through the perfume section of a department store can attest that the perfume sellers provide ample olfactory evidence about their wares.

Still more examples of such information provision include schools that hold open houses or allow prospective students and their families to set up guided tours, and even companies that enact generous return policies for their products. Yet this information comes at a cost– the consumer must expend energy, time, and money to visit each firm, and even returning a good via mail requires at least a modicum of time and effort, even if there is no monetary cost to a return. As a result of this cost, the consumer must take into account the amount of information that she will obtain by visiting each particular firm when she decides which firms to visit and when, eventually, to stop and purchase a product. In turn, this affects the information provided by the firms, who recognize how their information provision policies determine the consumer's search policy, and hence the likelihood of her eventually buying the product from them.

Inspired by such observations, this study tackles a fundamental question: how do search frictions shape information provision by competing firms? We investigate a single-product oligopoly setting in which n (≥ 2) firms compete by choosing how much information a representative consumer will obtain about their respective products upon her visit to their stores. We abstract away from price competition[1] and focus on the information provision problem of the firms. Each firm has complete freedom over how much information about its product's quality (or alternatively how much information about her own value for the product) the consumer will obtain during her visit.

1.1 Weitzman Search

Formally, the model is an extension of the classic "Weitzman" search problem, as described in Weitzman (1979), in which the objects to be searched (the firms) are themselves endowed with agency. In particular, each firm commits *ex-ante* to a signal or experiment, the results of which the consumer observes when visiting that firm. Crucially, the consumer observes the experiment (but not its realization) chosen by each firm before commencing search.[2]

Once the information structures have been chosen, the consumer faces a number of firms,

[1]This is justified if the competing firms are dealers for some product whose price is set by a central office. Such firms can control the level of information that they provide but not the price.

[2]The problem in Weitzman (1979) can be seen as the special case where the signal designed by each firm is fully revealing.

each offering some good whose random (expected) quality's distribution[3] is known to the consumer. Time is discrete and each period, the consumer may visit (alternatively, inspect) any firm that she chooses. She may visit only one firm per time interval and incurs a cost of $c \geq 0$ for such a visit. This is search with recall: having visited a firm, the consumer may always return to select that firm's product.

We assume the classic setup, in which the consumer must visit a firm before she can buy from it.[4] Consequently, the consumer has two decisions to make: she must decide in which order to visit the firms and when to stop and select one of them. From Weitzman (1979), the solution to this problem takes a simple form. Given its choice of signal, each firm is assigned a value, a reservation price characteristic to that firm, and the consumer simply inspects the highest value firm first, then the next highest value firm, etc. If at any time the consumer has already investigated a firm whose product has a posterior (expected) quality that is higher than the highest value of the remaining firms, the consumer stops the search and selects that firm. Naturally, each firm wants to be selected and chooses a signal that maximizes the probability of that occurrence, given an optimal search procedure by the consumer.

For simplicity, we restrict attention to the scenario in which each of the n firms has a good of either high or low (random) quality[5], where the probability of high quality, μ, is the same for each firm. Without loss of generality, we normalize the values of the low and high quality product to the risk-neutral consumer as 0 and 1, so μ is the expected value of the good's quality. The problem of experiment design for the firms is equivalent to one where each firm chooses any distribution supported on a subset of $[0, 1]$ such that expected value of its good's quality under this distribution is μ.

In the ensuing analysis we distinguish between the frictionless case and the case with frictions. That is, in the general model of Weitzman search, the consumer must pay some cost $c \geq 0$ to inspect a firm's product, and is impatient, discounting the future by some

[3]When the consumer arrives at a firm she observes a signal realization, and updates her belief about the quality of the product. The firm's choice of signal is equivalent to a choice of distribution of posterior expected qualities.

[4]Perhaps surprisingly, this assumption is non-trivial, as illustrated in the recent paper by Doval (2018).

[5]Alternatively, this is the consumer's match value with the product. We use quality throughout in order to easily differentiate between it and the reservation value assigned to the firm in the consumer's search problem.

positive $\beta \leq 1$. Accordingly, the frictionless case refers to the situation where $\beta = 1$ and $c = 0$.[6] Without frictions, a unique fair[7] symmetric equilibrium in pure strategies exists; and crucially, for any finite n the optimal experiment design is **not** fully informative. However, as the number of firms becomes infinitely large, the equilibrium converges to one in which each firm uses a fully informative signal. This can be thought of as the *Perfect Competition* level of information provision.

If instead there are frictions–there is some nonzero cost of visiting a firm and/or the consumer is relatively impatient–the equilibrium structure is different. Because of the search cost, the order of whom to visit becomes paramount, and the consumer proceeds optimally via an index policy. Crucially, the highest index (or reservation value) is uniquely induced by the distribution corresponding to a fully informative signal. Thus, if the expected quality of the product is sufficiently high, $\mu > \bar{\mu} := 1 - (1/n)^{1/(n-1)}$, the unique symmetric equilibrium is for each firm to choose a fully informative signal (Proposition 4.1). There is no profitable deviation from full information since any other distribution would ensure that the deviating firm be visited last. Then, since the frequency of high quality products under the prior is so high, the probability of the deviating firm being visited by the consumer, and hence being purchased from, is also low.

Accordingly, if the firms have a high average quality ($\mu > \bar{\mu}$), for any finite number of firms, n, a small cost $c > 0$ is actually welfare improving for the consumer and leads to the perfect competition level of information provision. A search cost and a high mean force the firms to compete with information, and that additional competition benefits the consumer.

On the other hand, if the firms do not have a high average quality then there are no symmetric equilibria in pure strategies. Now, a firm can deviate profitably from providing full information by being completely uninformative: even though it will be visited last–indeed, it will only be visited if every other firm is low quality–if it is visited it will certainly be selected and the low expected quality ensures that this event is sufficiently frequent.

[6]We could also think of this case as one in which the consumer sees all of the signals simultaneously. This interpretation would also lend support to our focus on fair subgame perfect equilibria.

[7]Defined formally *infra*. Essentially, the fair assumption imposes that the consumer randomize uniformly when she is indifferent.

Instead, in the low and medium average quality cases there are symmetric mixed strategy equilibria. In any such equilibrium the firms mix over a continuum of information levels, which distribution over experiments, crucially, has no atoms on any set of posteriors that induce the same reservation value except possibly on the maximum value (corresponding uniquely to full information). If the average quality is sufficiently low, $\mu < \underline{\mu} := 1/n$, then the distribution is completely atomless and the firms never provide full information. Conversely, in the medium average quality case ($\bar{\mu} > \mu > \underline{\mu}$) the firms are fully informative with strictly positive probability.

We explicitly construct mixed strategy equilibria in both the low and medium average quality cases. In both cases, the firms mix over binary distributions in which the low and high realizations are decreasing and increasing, respectively, in the reservation values that they induce. As a result, when the consumer commences on her search, she is faced with a collection of firms whose signal choices can be ranked in the Blackwell order. Indeed, the distributions over posterior (expected) qualities will be nested, like a matryoshka doll. Moreover, this equilibrium has the interesting property that despite this being a situation of search with recall, the consumer does not utilize this feature (at least not on the equilibrium path). She never returns to a firm to select its product, and if a firm is the last one to be visited, it is selected for sure.

In summation, this paper makes several observations about the effect of search frictions on information provision. First, when goods have a high expected quality, the competitive pressures engendered by the search cost have bite and encourage information provision. Second, when goods are of low average quality, we should expect a great deal of heterogeneity in the information provided. Some firms may provide a significant amount of information while others may provide very little, but we should not expect full information to be proffered. Third, if goods are of middling expected quality, again we should expect heterogeneity, though one or more of the firms may be fully informative. Fourth, all else equal, large markets encourage information provision (we find that no matter the expected quality of the product, μ, if the number of firms, n, is sufficiently large, the unique symmetric equilibrium is for the firms to provide full information).

In addition, this paper provides a novel channel through which coarse information provision by firms arises, which facilitates simple search protocols by consumers. Even though there are no information processing costs for the consumer, we find here that there are equilibria in which the firms provide coarse information. Namely, at each firm, there is either a high realization, following which the consumer finds it optimal to stop immediately and select that firm; or a low realization, after which the consumer prefers to continue her search. Moreover, the (possibly cognitively demanding) index policy is not needed on-path–the consumer simply visits the firms in order of highest possible maximal posterior (expected) quality to lowest. Accordingly, advertising policies whose simplicity is ostensibly the conscious choice of firms accommodating limited information processing abilities by their consumers, may instead correspond to equilibria of an information provision game like this one, where the firms (and consumers) have no such concerns or issues.

This study is also the first to look at a new aspect of information design. The idea that information structures can be designed in order to persuade is well established in the literature, and recent work has illustrated how information structures can be designed in order to provide incentives to an agent (or multiple agents) as well. Here, however, information structures are designed in order to entice. That is, the information design in this paper affects the consumer's decisions not only through what she learns, but also what she *will* learn. The information structures here shape the consumer's decisions before she even learns anything.

This section concludes with a brief discussion of related work. Section 2 describes the model, Section 3 characterizes equilibria of the game in the frictionless case, Section 4 does the same in the case with frictions, Section 5 looks at the effect of frictions on the consumer's welfare, Section 6 establishes the robustness of some of the results to alternate conditions, and Section 7 concludes. All proofs are left to the appendices, unless stated otherwise.

1.2 Related Work

Another paper that looks at information design in a consumer search setting is Board and Lu (2018). There, the authors examine the case of a collection of sellers competing to persuade a prospective buyer to purchase one of their products. Each seller has the same set of products to sell and designs an experiment in order to sway the buyer's opinion on the product. In contrast to our paper, buyers pick a seller at random; and moreover, the state of the world upon which the searchers' utility depends is common. Here, the information provided by each firm provides the consumer with no information about the contents of any of the other firms.

There are a number of papers that derive results that correspond to various simplifications of the *frictionless* search problem. In Spiegler (2006), the unique equilibrium is essentially isomorphic to the specific case in the frictionless model when the mean is fixed at 1/2. Boleslavsky and Cotton (2015) and Albrecht (2017) each derive results that characterize the the two-player solution to this problem of (frictionless) competitive information provision. Other papers that look at competitive information provision include Koessler et al. (2017), Boleslavsky and Cotton (2018), and Au and Kawai (2019).

Special mention is due to Au and Kawai (2020), which establishes the unique pure strategy equilibrium for the senders in a case identical to this paper's frictionless case. The frictionless results in this paper were derived independently of those in Au and Kawai (2020), using different techniques. Whitmeyer (2018) looks at a two-player dynamic version of the frictionless case, in which two firms provide information about their respective products in each period to a short-lived consumer in order to secure her favor.

Another similar paper is Garcia (2018), which goes beyond the basic competitive persuasion environment and looks at the optimal policies for sellers in a frictionless search market who can compete both on information provision and price. Similarly, Hwang et al. (2018) look at information design in a search market between sellers who can compete on information provision and price. They show that the advertising component of the sellers' equilibrium strategies must have a particular form: the distribution of the largest order statistic must be convex and have an (alternating) piece-wise linear form, which result generalizes the frictionless case in this paper (as well as the results in Albrecht (2017), Boleslavsky and Cotton (2018), and Au and Kawai (2020)). Hwang et al. (2018) contains no search frictions; however, which are a crucial component to this work.

Note that this idea of competitive persuasion concerns a different sort of competition than that featured in Gentzkow and Kamenica (2017a,b), or Li and Norman (2017). In those problems, the persuaders inhabit the same common state and independently choose experiments to convey information about this common state. In contrast, the firms here each present information about their own i.i.d. draws, and not some common state. In addition, this idea of information provision as the choice of a Blackwell experiment is also used in, among others, Gentzkow and Kamenica (2016); Kolotilin et al. (2017); and Perez-Richet and Skreta (2017).

There are several papers in the search literature that bear mentioning as well. In particular, the vein of research that focuses on *obfuscation* is relevant. Two such papers are Ellison and Wolitzky (2012) and Ellison and Ellison (2009). In the first, the authors extend the model of Stahl (1989) by allowing firms to choose the length of time it takes for consumers to learn its price. Allowing for such delays hurts consumers, since obfuscation leads to longer search times and higher prices. Ellison and Ellison, in turn, provide empirical evidence suggesting that as technology has made price search easier for consumers, firms have responded by taking actions that make price search more difficult.

More recently, Gamp and Krähmer (2017) examine a scenario in which sellers can dupe naive consumers into buying products that are lemons. As in this model (though through a different mechanism), search frictions can be beneficial to consumers. Choi et al. (2018) incorporate price competition into a model of Weitzman search and show, among other things, that a reduction in search cost can lead to an increase in price. Perez-Richet (2012) considers a model in which candidates can choose whether to disclose information in order to convince a decision maker to adopt their product. Armstrong (2017) looks at the Weitzman search problem in which sellers compete on price. In particular, he observes that when prices are posted, search frictions are likely to beget price competition due to the importance of being visited early. However, the overall effect on consumer welfare is not obvious: the benefit to consumers of lower prices may be outweighed by the search costs. Petrikaitė (2017) modifies this problem slightly in a duopoly setting by allowing the consumer to buy a seller's product without visiting and paying the inspection cost (as in Doval (2018)). This has a surprising effect: in contrast to the standard Weitzman setting, even though prices are posted, *lower* search costs may beget price competition. However, Proposition 6.1 in this paper illustrates that such a modification to the consumer's search problem does not alter the effect of search frictions on information provision.

Finally, there are a number of papers that look at the ramifications of a consumer having limited information about her valuation for a product. Those include Roesler and Szentes (2017), Condorelli and Szentes (2020), and Yang (2019) in versions of a bilateral trade model; Choi et al. (2019) in a search good (where true values are apparent upon inspection instead of through consumption) environment; Dogan and Hu (2018) in a (random) sequential search setting; and Armstrong and Zhou (2019) in an oligopoly setting.

2 The Model

This game consists of $n + 1$ players; one consumer, C, and n *ex-ante* identical single-product firms indexed by i. Each firm's product has a random quality (or match value) to the consumer of either 0 or 1, and μ is the probability of the quality being 1. We call this prior G_0 and impose that each firm's quality is independent. In addition, an inspection cost, $c \geq 0$, is associated with each firm.[8] The consumer's utility function $u_C(\cdot)$ is linear in a product's quality.

Each firm has a compact metric space of signal realizations S. Prior to learning its quality (match value with the consumer), each firm chooses a signal, Borel measurable function $\pi_i : \{0, 1\} \longrightarrow \Delta(S)$, and has the power of full commitment to the signal. For any firm i, each signal realization s_i leads to a posterior distribution $G_{s_i} \in \Delta(\{0, 1\})$ with the corresponding posterior mean μ_{s_i}. Accordingly, the signal leads to a distribution over posterior means, $\tau_i \in \Delta(\mu_{s_i})$.

We impose that the consumer's reservation value or outside option is 0, and that the search cost, c, is sufficiently low that the consumer finds it optimal to search i.e. $\mu > c$. The consumer, before embarking on her search, observes the signal (or experiment) chosen by each of the firms. Accordingly, she knows how informative each firm has decided to be about its product before she decides whom to visit. When the consumer visits a firm, she observes the signal realization, and updates her prior as to the posterior (expected) quality of the product offered by the firm.

[8] In Weitzman (1979), there is both a monetary cost, c, and a time cost, β, to inspecting a firm's product. Because a decrease in β has the analogous effect to an increase in c (and vice versa), β is redundant. Accordingly, we set $\beta = 1$ throughout, which simplifies notation and exposition.

Remark 2.1. Each firm's problem is equivalent to one in which each firm chooses a distribution F_i supported on $[0, 1]$ with mean μ.

We stipulate firm i's utility in the game to be the probability that it is the firm selected when the consumer chooses to stop searching and select a firm. Accordingly, this game is a constant sum game: the payoffs of all the firms sum to 1.

Definition 2.2. Given a Bernoulli prior with mean μ, the set of **Pure Strategies** for each firm is the set of distributions supported on $[0, 1]$ with mean μ. That is, a pure strategy for firm i consists of a choice of distribution $F_i \in M(\mu)$.

The set of mixed strategies is defined as expected:

Definition 2.3. The set of **Mixed Strategies**, Σ_i, is the set of all Borel probability distributions over the pure strategies of firm i:

$$\Sigma_i = \left\{ \sigma_i \, : \, M(\mu) \longrightarrow [0, 1]^{|M(\mu)|} \quad \Big| \sum_{F_i \in M(\mu)} \sigma_i(F_i) = 1 \right\}$$

A typical element $\sigma_i \in \Sigma_i$ is a Borel probability distribution over the (compact and equipped with a metric) set $M(\mu)$ of firm i's pure strategies.

The timing of the game is

1. Each firm simultaneously chooses a mixed strategy $\sigma_i \in \Sigma_i$.

2. For each firm, the outcome of the randomization over pure strategies is realized.

3. The consumer observes each distribution over qualities F_i and chooses an optimal *policy*, η, which consists of a *selection rule* dictating the order in which she should inspect the firms' products, and a *stopping rule* deciding when she should stop searching.

2.1 The Consumer's Problem

Given the choice of distributions by each of the firms, the consumer is faced with n firms, each containing a posterior (expected) quality X_i distributed according to a probability distribution function F_i, which is an endogenous choice of firm i. The seminal result of Weitzman (1979) was to show that each firm can be fully characterized by its *reservation price*:

Definition 2.4. The reservation price of a firm i is z_{F_i}, where z_{F_i} is defined as

$$z_{F_i} = -c + \int_{-\infty}^{z_{F_i}} z_{F_i} dF_i + \int_{z_{F_i}}^{\infty} x_i dF_i$$

Given this definition, Weitzman proved the following proposition:

Proposition 2.5. *An optimal (pure strategy) policy, η, or "Pandora's Rule" (Weitzman (1979)), is completely characterized by the following two rules:*

1. ***Selection Rule:*** *If a firm is to be visited and examined, it should be the unvisited firm with the highest reservation price.*

2. ***Stopping Rule:*** *Search should be stopped whenever the maximum reservation price of the unvisited firms is lower than the maximum sampled reward, w.*

Note that at each point in the consumer's search we can partition the set of firms into visited and unvisited firms, O and U, respectively. Hence, w is given formally as

$$w = \max_{i \in O \cup 0} x_i$$

Let $\mathbf{F}$ be the Cartesian product, $\mathbf{F} := \prod_i F_i$. Given a vector of mixed strategies, $\sigma_{-i} \in \Sigma_{-i}$, by the other firms and an optimal policy, $\eta(\mathbf{F})$, by the consumer, firm i's expected payoff from choosing mixed strategy, σ_i, is

$$\mathbb{E}\left[u_i\left(\sigma_i, \sigma_{-i}; \eta(\mathbf{F})\right)\right] = \mathbb{P}\left(\text{Firm} \quad i \quad \text{is chosen} \quad | \, \sigma_i, \sigma_{-i}, \eta(\mathbf{F})\right)$$

We also impose the following notion of fairness: if the consumer is indifferent about whom to visit next, she randomizes fairly over the firms responsible for the indifference. Formally, we define U^* as the set of firms that induce the maximal z value of all of the firms in U:

$$U^* := \left\{ i \in U \, : \, z_i \geq z_j \quad \forall j \in U \right\}$$

Then,

Definition 2.6. A policy, η, is **Fair** if the consumer visits each member of U^* next with equal probability.

Accordingly, in a Fair Subgame Perfect Equilibrium, given the consumer's optimal fair policy, η, each firm must choose a signal in order to maximize the chance that it is chosen by the consumer, given the optimal signals by the other firms. Formally,

Definition 2.7. A mixed strategy **Fair Subgame Perfect Equilibrium** (FSPE) consists of a policy, η^*, for the consumer and a vector of strategies for the firms, Σ^*, such that

1. The consumer chooses a **Fair** optimal policy:

   (a) C's optimal policy, $\eta^*(\mathbf{F})$, satisfies $u_C(\eta^*(\mathbf{F})) \geq u_C(\eta'(\mathbf{F})) \;\; \forall \;\; \eta'$; and

   (b) C's optimal policy is **Fair**.

2. For each firm i, $\mathbb{E}[u_i(\sigma_i^*, \sigma_{-i}; \eta^*)] \geq \mathbb{E}[u_i(\sigma_i^{'}, \sigma_{-i}; \eta^*)] \;\; \forall \;\; \sigma_i^{'} \in \Sigma_i$.

Having fully established what the consumer will do upon seeing the signal choices by the firms, we can work backward and focus our attention on the strategic interaction between the firms.[9]

Because the game played by the firms is discontinuous, the existence of equilibria isn't immediately evident. However, our subsequent analysis provides a constructive proof of the following proposition:

Proposition 2.8. *A fair subgame perfect equilibrium in mixed strategies always exists.*

Note that there is an important distinction between the frictionless case ($c = 0$) and the case with frictions ($c > 0$). Namely, in the frictionless case the sets of pure strategies and mixed strategies are the same (or equivalent). However, as the following example illustrates, this is not the case when there are search frictions.

Example 2.9. Let $\mu = 2/5$, and for simplicity let the number of firms, n, equal 2. Let $c = 1/10$, and consider the following three pure strategies:

1. A: 2/5 with probability 1;

[9]Henceforth we use FSPE, SPE, equilibrium, and subgame perfect equilibrium interchangeably–to refer to this idea of FSPE.

2. B: 1 with probability 2/5, and 0 with probability 3/5;

3. D: 1 with probability 1/5, 2/5 with probability 1/2, and 0 with probability 3/10.

Consider also the mixed strategy σ^{AB}, where a firm plays pure strategies A and B with probability 1/2 each. In the frictionless case, strategies σ^{AB} and D are equivalent. However, as we discover here, with frictions, this is no longer the case.

The corresponding reservation values for the three pure strategies, respectively, are: 3/10, 3/4, and 1/2. Suppose that firm 2 chooses strategy A and firm 1, strategy D. Firm 1 is chosen first, and with probability 7/10, the quality realization is greater than $z_A = 3/10$. Thus, firm 1's payoff under this vector of strategies is 7/10. If both firms choose strategy A, then each will obtain 1/2. Finally, if firm 2 chooses A and firm 1, B, firm 1 will obtain a payoff of 2/5. Should firm 2 play A and firm 1 mix equally over A and B, firm 1's payoff would be $9/20 < 7/10$. Thus, σ^{AB} and D are not equivalent.

Hence, when there are no frictions, a symmetric pure strategy equilibrium in pure strategies exists. As we will shortly discover, with frictions, outside of the high average quality case, there are no symmetric pure strategy equilibria.

3 Frictionless Search

We begin the analysis with the case where the inspection cost, c, is equal to 0. We call this frictionless search. Alternatively, this can be thought of the situation in which the consumer observes all of the signal realizations simultaneously. Indeed, this interpretation lends itself naturally to the focus on *fair* equilibria. Since $c = 0$, this scenario is equivalent to one in which a mean $\mu \in [0, 1]$ is fixed, and the firms each choose a random variable distributed on $[0, 1]$ with the given mean. The winner of the game is the firm whose random variable has the highest realization.

Define

$$\underline{\mu} := \frac{1}{n}$$

Then,

Theorem 3.1. *1. If $\mu \geq \underline{\mu}$ then the unique symmetric pure strategy equilibrium is for each firm to choose distribution F^*, defined as*

$$F^*(x) = \begin{cases} (1-a)\left(\frac{x}{s}\right)^{\frac{1}{n-1}}, & 0 \leq x < s \\ (1-a), & s \leq x < 1 \\ 1, & 1 = x \end{cases}$$

where

$$a = \mu - \mu(1-a)^n, \quad \text{and} \quad s = n\mu(1-a)^{n-1}$$

2. If $\mu \leq \underline{\mu}$ then the unique symmetric pure strategy equilibrium is for each firm to choose distribution F^*, defined as

$$F^*(x) = \left(\frac{x}{n\mu}\right)^{\frac{1}{n-1}}, \quad 0 \leq x \leq n\mu$$

Proof. Because this result, or variants thereof, was derived independently in several other papers–Spiegler (2006), Boleslavsky and Cotton (2015), Albrecht (2017), and Au and Kawai (2020)–the detailed proof is left to the Online Appendix.[10] However, we briefly explore a sketch of the proof here, because the techniques used to derive the result (in particular uniqueness) may be of use in other information design problems.

It is straightforward to verify directly that no firm can deviate profitably from F^* when the others all choose it. Establishment of uniqueness proceeds in two steps. First, through reasoning similar to that in all-pay auctions, in any symmetric equilibrium the distributions chosen by the firms cannot have mass points except possibly at the top (on 1). Moreover, if the mean is sufficiently high, then any symmetric equilibrium must have a point mass on 1, since otherwise a firm could deviate profitably by choosing a fully informative signal. Against firms choosing strategies without point masses on 1, a firm playing the prior would be selected for sure after the high realization, and so if the high realization is sufficiently likely, the other firms need to take steps to thwart this.

In the second step, we merely solve for the equilibrium strategy directly, using optimal control techniques. We fix a vector of strategies, H, for the $n-1$ other firms: $H := F_j^{n-1}$.

[10] See also the (now defunct) working paper Hulko and Whitmeyer (2017).

Given H, we define the functional $J[f_i]$ as

$$J[f_i] = \int_t^s f_i(x)H(x)dx - \lambda_0 \left[\int_t^s f_i(x)dx - (1-a)\right] - \lambda \left[\int_t^s x f_i(x)dx - \mu + a\right]$$

where the first constraint ensures the distribution satisfies Kolmogorov's second axiom, and the second constraint guarantees that the expectation is μ (and a is the size of the point mass on 1). We appeal to the fundamental lemma of calculus of variations and derive the Euler-Lagrange equation, which is a necessary condition for an extremum. Our proposed equilibrium is the unique (symmetric) solution. We finish by generating some smaller auxiliary results that are needed in order to pin down the support of F_i. ∎

For $\mu \leq \underline{\mu}$, the unique (pure strategy) equilibrium strategies are such that the distribution of the maximum of realized values of the $n-1$ other firms is the uniform distribution on $[0, n\mu]$. If $\mu \geq \underline{\mu}$ then each individual firm still faces a distribution generated by the $n-1$ other firms on some interval $[0, s]$ but now each firm also puts a point mass of weight $a \geq 0$ on 1.

Unsurprisingly, as illustrated by the next result, competition has a clear effect on information provision in the frictionless case.

Corollary 3.2. *Let $\mu > \underline{\mu}$. If the number of firms is increased, the weight placed on* 1 *in the symmetric equilibrium must increase. That is, a is strictly increasing in n.*

In the limit, as the number of firms, n, becomes infinitely large, the weight on 1 *converges to μ. That is, the equilibrium distribution converges to a distribution with support on two points,* 1 *and* 0.

This proof is also left to the Online Appendix and follows through direct computation.

4 Search with Frictions

We turn our attention to the case where $c > 0$. We begin by establishing sufficient conditions for there to be a symmetric equilibrium in pure strategies.

Recall the cutoffs between the low and medium average quality, and medium and high average quality cases, $\underline{\mu}$ and $\bar{\mu}$, respectively:

$$\underline{\mu} := \frac{1}{n}, \quad \text{and} \quad \bar{\mu} := 1 - \left(\frac{1}{n}\right)^{\frac{1}{n-1}}$$

In addition, we use the following labels for the maximal and minimal reservation values. They are, respectively,

$$z_{G_0} := \frac{\mu - c}{\mu}, \quad \text{and} \quad \underaccent{\tilde}{z} := \mu - c$$

As Lemma 4.2 illustrates, the maximal reservation value, z_{G_0}, is induced uniquely by the Bernoulli distribution over qualities corresponding to a fully informative signal. The minimal reservation value, $\underaccent{\tilde}{z}$, is induced by any distribution whose support lies (weakly) above $\mu - c$, of which collection one distribution is the (degenerate) point mass on μ corresponding to a fully *uninformative* signal. Then,

Proposition 4.1 (High Average Quality). *If $\mu \geq \bar{\mu}$ there is a symmetric equilibrium in pure strategies where each firm chooses a fully revealing signal. If this inequality is strict, or if $n \geq 3$, this equilibrium is unique among symmetric equilibria (pure or mixed). If $n = 2$ and $\mu = 1/2$ then there exists a continuum of mixed strategy equilibria described in Proposition 4.7 where the point mass on z_{G_0}, γ, takes any value in* $[0, 1]$.

For any fixed $\mu \in (0, 1)$, there is some N such that if the number of firms, n, is greater than or equal to N, there exists a unique symmetric equilibrium in pure strategies, where each firm chooses a fully informative signal.

Proof. We establish the result through the following two lemmata:

Lemma 4.2. *The reservation value assigned by C to a firm with a fully revealing signal is strictly higher than the reservation value assigned to a firm with any other signal. Equivalently, if $F_i \in M(\mu)$ and $F_i \neq G_0$ then $z_{G_0} > z_{F_i}$.*

For a general prior, $z_{G_0} \geq z_{F_i}$ for any F_i that is a mean-preserving contraction of G_0; which is intuitively clear[11] since G_0 is more informative in the Blackwell sense than any F_i. However, the fact that the reservation value assigned to the prior is *strictly* higher than for any other garbling relies on the binary prior assumption. Indeed, it does not hold in general for a prior with any other support.

Lemma 4.3. *If $\mu \geq \bar{\mu}$, then a symmetric (pure strategy) vector of distributions is an equilibrium if and only if each distribution induces the maximal reservation value, z_{G_0}.*

[11] A formal proof is available upon request.

Combining the two lemmata yields Proposition 4.1. Coexistence of the full information equilibrium and the continuum of mixed strategy equilibria at $\mu = 1/2$ and $n = 2$ is a knife-edge case engendered by the fact that when $\mu = 1/2$, $\underline{\mu} = \bar{\mu}$. ■

If the mean is sufficiently high, then it becomes paramount for a firm to choose an experiment that entices the consumer into visiting it first. Since there are search costs, Pandora's rule dictates that she visit more informative firms first. Hence, in any proposed symmetric equilibrium other than the fully informative one, a firm can deviate by choosing a distribution with a slightly higher reservation value, thereby ensuring that it is the first firm visited. As a result of this incentive, there is an unraveling upwards to the distribution that induces the maximal reservation value, which is uniquely the prior distribution. Because the mean is so high, there can be no profitable deviation from the fully informative equilibrium. Crucially, any firm that deviates from the fully informative equilibrium will guarantee that it is visited last. Then, because the mean–and equivalently the chance of a high realization under the fully informative signal–are so high, even though a deviating firm can ensure that if it is visited it will be selected, the likelihood that the consumer ever visits that firm is too low for the deviation to be profitable.

If, however, the mean, μ, is strictly *below* the cutoff ($\mu < \bar{\mu}$) then this logic fails, since it is insufficiently likely that the firm is of high quality. As in the high mean case, because of the incentive to be visited first, there cannot be a symmetric pure strategy equilibrium in which firms are anything less than fully informative. However, because the mean is so low, there is no symmetric pure strategy equilibrium in which each firm is fully informative. There, the likelihood that the deviating firm will be visited is sufficiently high to ensure that the deviation is profitable. Indeed, we have

Proposition 4.4. *If $\mu < \bar{\mu}$ there are no symmetric equilibria in pure strategies.*

Proof. We sketch the proof (with formal detail left to Appendix A.3).

First, there can be no symmetric pure strategy equilibrium in which firms play distributions with any $z_{F_i} < z_{G_0}$. A firm can always deviate by choosing a distribution that is almost identical but which induces a slightly higher z. In doing so, the deviating firm ensures that it will be inspected first, securing itself a first-order increase in its payoff (whereas any potential decrease in payoff resulting from the change in distribution will be second-order).

Second, because the mean is so low, there can be no equilibrium in which each firm chooses the fully informative signal, since a firm could deviate profitably to, say, a fully *uninformative* signal, and achieve a higher payoff. ■

We turn our attention to finding a symmetric mixed strategy equilibrium. We say that there is an induced mass point on a reservation value z if there is a set of distributions that induce z that is played with positive probability.

Lemma 4.5. *There are no symmetric mixed strategy equilibria in which there is an induced mass point $a > 0$ on any reservation value $z \neq z_{G_0}$.*

Proof. Recall that z_{G_0} is the reservation value induced uniquely by a fully informative signal. The intuition behind this proof is similar to that for Proposition 4.4. Any firm i can deviate profitably by instead inducing a mass point on a slightly higher reservation value. In a sense this can be thought of an identical mixed strategy where the "ties are broken in firm i's favor".

The detailed proof is included in Appendix A.4. ■

This result arises as a consequence of the same sort of logic that dictates that firms choose atomless distributions over prices in situations where their competitors can undercut, see e.g. Stahl (1989). In our scenario, if there were a point mass on any reservation value other than the maximal reservation value, another firm could "overcut" with information, and discretely increase its payoff.

Furthermore, because of the atomless (except possibly at the top) nature of the induced distribution over reservation values, z, it is without loss of generality to suppose that each z corresponds uniquely to some distribution over qualities. *Viz*, there are not multiple distributions over qualities in the support of a firm's mixed strategy that correspond to the same reservation value z.

Precise characterization of the mixed strategy equilibrium is a challenge. The utility of a firm as a function of the vector of mixed strategies is quite complicated, due to the nature of the consumer's stopping procedure. In a working paper, Au (2018) establishes some other properties that the mixed strategy must have. In particular, Au shows that the induced distribution over reservation values must have support on $[\underline{z}, \bar{z}] \cup z_{G_0}$ or $[\underline{z}, \bar{z}]$ for some $\bar{z} \leq z_{G_0}$ (Corollary 1, Au (2018)). This corollary and Lemma 4.5 are all we need to explicitly characterize an equilibrium.

We look for a symmetric mixed strategy equilibrium that induces an atomless distribution, $\Phi(z) := \mathbb{P}(Z \leq z)$, over reservation values (we can think of each firm choosing the distribution of some random variable Z) with support on

$$[\underline{z}, \bar{z}]$$

where $\bar{z} \leq z_{G_0}$. As noted several paragraphs prior, because Φ is atomless it is without loss of generality to impose that each distribution over qualities corresponding to each z in the support of Φ is unique. In addition, we make the *ansatz* that each z is induced by a binary distribution $a(z) \geq z$ with probability $p(z)$ and $b(z) \leq z$ with probability $(1 - p(z))$. Furthermore, we also guess that b is strictly increasing in z, which greatly simplifies the analysis (though this is a little reminiscent of the quip about the drunk, who lost his keys at home, looking for them outside under a streetlight because "the light is better there").

As we discover, in such an equilibrium, $b(\bar{z})$ must equal 0 and $b(\underline{z})$ must equal $\underline{z}(= \mu - c)$. Moreover, $\bar{z} = n(\mu - c)$, since otherwise a firm could deviate by choosing some z in between $\bar{z}$ and $n(\mu - c)$, guaranteeing both that it would be visited first and have a high realization with a probability strictly greater than $1/n$.

In the low average quality case ($\mu \leq \underline{\mu}$), such an equilibrium is always feasible, and, leaving the detailed derivation to Appendix B, we obtain the following equilibrium.

Proposition 4.6 (Low Average Quality). *Let $\mu \leq \underline{\mu}$. Then there is a symmetric mixed strategy equilibrium in which firms randomize over a one-dimensional class of distributions with binary support. The class is indexed by z (the induced reservation value), which is distributed according to the atomless distribution $\Phi(z)$ on $[\underline{z}, \bar{z}]$. Each binary distribution in support of the mixed strategy consists of two points, $a(z)$ with probability $p(z)$ and $b(z)$ with probability $1 - p(z)$.*

Thus, the equilibrium is described by four functions $a(z)$, $b(z)$, $p(z)$ and $\Phi(z)$, where $b(z)$ is defined implicitly as

$$b^{\frac{n}{n-1}} - \bar{z} b^{\frac{1}{n-1}} = z^{\frac{n}{n-1}} - \bar{z} z^{\frac{1}{n-1}} \tag{$\star$1}$$

and

$$\Phi(z) := \mathbb{P}(Z \leq z) = \left(\frac{1}{\bar{z}}\right)^{\frac{1}{n-1}} \left\{ z^{\frac{1}{n-1}} - b(z)^{\frac{1}{n-1}} \right\} \tag{$\star$2}$$

$$a(z)p(z) = \mu - b(z)\left(1 - p(z)\right) \tag{$\star$3}$$

and

$$zp(z) = a(z)p(z) - c \tag{$\star$4}$$

where

$$\underline{z} = \mu - c, \quad \bar{z} = n\left(\mu - c\right), \quad b(\underline{z}) = \underline{z} = \mu - c, \quad b\left(\bar{z}\right) = 0, \quad p\left(\underline{z}\right) = \frac{1}{2}, \quad \textit{and} \quad p\left(\bar{z}\right) = \frac{1}{n}$$

a is monotonically increasing in z, and b and p are monotonically decreasing in z.

Note that the last two equations hold by construction: Equation $\star$3 is implied by the fact that the distribution that induces each z is binary with mean μ; and Equation $\star$4 merely follows from the definition of z.

Importantly, because the firms are mixing over distributions, there are two points in the game when some randomness is resolved. The first point is initially, when the firms simultaneously randomize through Φ over binary distributions and the reservation values they induce. After each firm's binary distribution over qualities is realized, the consumer observes these realized binary distributions before assigning each firm a reservation value and deciding upon her search order. The second point when some randomness is resolved is during the search itself, when the consumer visits a firm and observes the draw from that firm's binary distribution.

This equilibrium has several interesting properties. Once the initial randomness from the firms mixing is resolved, the consumer will be faced with n firms, each with unique binary distributions that nest within each other, like a matryoshka doll. Figure 1 illustrates one realization of the distributions over qualities that the searcher faces for three firms with low expected quality. She visits Firm 1 (green) first and stops at the high realization. Otherwise, she visits Firm 2 (blue) next, where again she stops at the high realization. Should the low value there realize, she visits Firm 3 (red) and selects it no matter the realization.

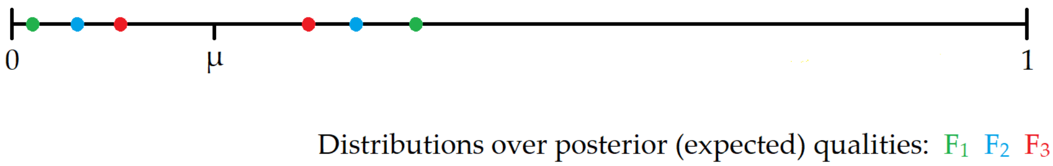

Figure 1: A Realized Search Problem for the Consumer, $n = 3$, $\mu = 1/5$

Evidently, the experiments chosen by the firms can be ranked according to the Blackwell order, and, on path, the consumer searches them in order of their (Blackwell) informativeness. The consumer stops only if she observes the high realization at a firm. Otherwise, she continues her search, and selects the last firm no matter its realization. Though this is a search in which recall is allowed, the consumer never utilizes this, and *never* returns to a firm from which she had previously moved on.

We see that a scenario arises endogenously that allows for a much simpler (optimal) search protocol than Pandora's Rule. Namely, the consumer merely visits the firms in order of informativeness, and selects a firm if its high value is realized–indeed it is obvious that she should stop since she knows that she will not see a higher realization at any of the remaining firms. If she reaches the last firm, she selects that firm with certainty.

What happens in the medium average quality case, when $\underline{\mu} < \mu < \bar{\mu}$? In this scenario, the expected value of the product is high, but not sufficiently high to force the full information equilibrium. However, it is immediately clear that the equilibrium for the low average quality case is no longer feasible either. Recall that there, the firms choose atomless distributions on $[\underline{z}, \bar{z}]$ where $\bar{z}$ must equal $n(\mu - c)$ in order to thwart the incentive to deviate to a more informative distribution (with a higher reservation value) in order to be visited first. In the medium average quality case, this is impossible, since z_{G_0}, the maximal reservation value is strictly lower than $n(\mu - c)$.

Consequently, there can be no symmetric distribution that does not have an atom on the maximal reservation value, z_{G_0}. Without such a point mass, a firm could deviate profitably by providing full information. In addition, there is also a continuous portion of the distribution with support on $[\underline{z}, \bar{z}]$, where now $\bar{z}$ may not equal $n(\mu - c)$. We denote the size of the atom on z_{G_0} by γ and let $\Phi(z) := \mathbb{P}(Z \leq z)$ denote the continuous portion of the distribution, where

$$\Phi(\bar{z}) = 1 - \gamma, \quad \text{and} \quad \int_{\underline{z}}^{\bar{z}} w d\Phi(w) = \mu - \gamma$$

Again, we look for an equilibrium in which each z is induced by a binary distribution $a(z) \geq z$ with probability $p(z)$ and $b(z) \leq z$ with probability $(1 - p(z))$, where b is decreasing in z. As in the low average quality case, in such an equilibrium, $b(\bar{z})$ must equal 0 and $b(\underline{z})$ must equal $\underline{z}$. Explicitly,

Proposition 4.7 (Medium Average Quality). *Let $\underline{\mu} \leq \mu \leq \bar{\mu}$. Then there is a symmetric mixed strategy equilibrium in which firms randomize over a one-dimensional class of distributions with binary support. The class is indexed by z (the induced reservation value), and the distribution over values has an atomless portion, $\Phi(z)$, on $[\underline{z}, \bar{z}]$ and a point mass of size γ on z_{G_0}. Each binary distribution in support of the mixed strategy consists of $a(z)$ with probability $p(z)$ and $b(z)$ with probability $1 - p(z)$.*

Thus, the equilibrium is described by one constant, γ; and four functions, $a(z)$, $b(z)$, $p(z)$ and $\Phi(z)$; where $b(z)$ is defined implicitly as

$$\begin{aligned} &((1 - n\beta)\, b(z) + n(\mu - c)\beta)^{\frac{1}{n-1}} ((1 - n\beta)\, b(z) - (1 - \beta)\, n(\mu - c)) \\ &= ((1 - n\beta)\, z + n(\mu - c)\beta)^{\frac{1}{n-1}} ((1 - n\beta)\, z - (1 - \beta)\, n(\mu - c)) \end{aligned} \tag{$\star 1$}$$

and

$$\Phi(z) := \mathbb{P}(Z \leq z) = \left(\frac{1}{n(\mu - c)}\right)^{\frac{1}{n-1}} \left\{\xi(z)^{\frac{1}{n-1}} - [b(z)(1 - n\beta) + n(\mu - c)\beta]^{\frac{1}{n-1}}\right\} \tag{$\star 2$}$$

$$a(z)p(z) = \mu - b(z)(1 - p(z)) \tag{$\star 3$}$$

$$zp(z) = a(z)p(z) - c \tag{$\star 4$}$$

and

$$1 - \gamma = (1 - \mu\gamma)^n - ((1 - \mu)\gamma)^n \tag{$\star 5$}$$

where

$$\xi(z) := z(1 - n\beta) + n(\mu - c)\beta, \quad \alpha := (1 - \gamma\mu)^{n-1}, \quad \beta := (\gamma(1 - \mu))^{n-1}$$

and

$$\underline{z} = \mu - c, \quad \bar{z} = \frac{n(\mu - c)(\alpha - \beta)}{1 - n\beta}, \quad b(\underline{z}) = \underline{z}, \quad b(\bar{z}) = 0, \quad p(\underline{z}) = \frac{1}{2}, \quad and \quad p(\bar{z}) = \frac{1 - n\beta}{n(\alpha - \beta)}$$

This equilibrium is analogous qualitatively to the equilibrium in the low average quality case, with the exception of the mass point on z_{G_0}. Again, the realized distributions over qualities of the firms nest within each other and so the firms' signals can be ranked according to the Blackwell order. The consumer's search behavior is the same as well: she visits the firms in order of informativeness, stops only if she obtains the high realization (unless she's at the last firm), and never returns to a firm.

Note that Equation ⋆5 must hold in *any* equilibrium in which the firms place atoms on z_{G_0}. Moreover, the polynomial described in Equation ⋆5 has a root in the interval $(0, 1)$ if and only if $\underline{\mu} < \mu < \bar{\mu}$. Furthermore, γ defined implicitly as a function of μ in Equation ⋆5 is continuous and increasing in μ and equals 0 if $\mu = \underline{\mu}$ and 1 if $\mu = \bar{\mu}$. For all $\mu \in [0, 1]$ the size of the point mass on z_{G_0} in any symmetric equilibrium is increasing in μ: for $\mu \leq \underline{\mu}$ it is of weight 0, for $\underline{\mu} < \mu < \bar{\mu}$ it is positive and strictly increasing in μ, and for $\mu \geq \bar{\mu}$ it is of weight 1. These observations–in conjunction with Corollary 1 in Au (2018), Lemma 4.5, and Proposition 4.1–allow us to conclude the following theorem:

Theorem 4.8. *Any symmetric equilibrium must be as follows:*

1. ***High Average Quality.*** *$\mu > \bar{\mu}$, or $\mu \geq \bar{\mu}$ and $n \geq 3$: each firm provides full information.*
2. ***Medium Average Quality.*** *$\bar{\mu} \geq \mu \geq \underline{\mu}$: each firm chooses a distribution over reservation values that has an atom of size γ (defined implicitly in Equation ⋆5) on the maximal reservation value, z_{G_0} (corresponding to full information), and an atomless portion over reservation values with support on $[\underline{z}, \bar{z}]$.*
3. ***Low Average Quality.*** *If $\underline{\mu} > \mu$, or $\underline{\mu} \geq \mu$ and $n \geq 3$: each firm chooses an atomless distribution over reservation values with support on $[\underline{z}, \bar{z}]$, and chooses full information with probability 0.*

$\mu = 1/2$ and $n = 2$ constitutes a special (knife-edge) case. There, there is a continuum of mixed strategy equilibria in which the firms place any weight $\gamma \in [0, 1]$ on z_{G_0} and choose an atomless distribution over reservation values with support on $[\underline{z}, \bar{z}]$. Note that γ can equal 0 or 1.

The three regions are illustrated in Figure 2. As n increases, both $\underline{\mu}$ and $\bar{\mu}$ decrease, and in the limit they both go to 0. Thus, the larger the market, the lower the threshold needed for a product to qualify as high average quality.

As mentioned *supra*, Au (2018) establishes some other properties that the mixed strategy equilibrium must have and partially characterizes a mixed strategy equilibrium for the case where the consumer has an outside option that is strictly greater than the lowest possible reservation value. This begets an equilibrium in which the support of each firm's mixed strategy consists of binary distributions with the property that the low realization of any pure strategy in support is strictly less than the outside option. His assumption allows for a much simpler formulation of the firms' problem: it ensures that only the high realizations of a firm's pure strategy can possibly yield the firm a non-zero payoff. That is, if a firm does not have a high realization, then no matter when it is visited it will not be selected.

5 Consumer Welfare

In this section we delve into an important ramification of Proposition 4.1. Namely, we establish that if the product is high average quality, then an increase in search costs can increase the welfare of the consumer. We begin with consumer welfare in the frictionless case.

Let $\mu \geq \underline{\mu}$. If $c = 0$, then from Theorem 3.1, the unique symmetric equilibrium is for each firm to choose distribution F^*, defined as

$$F^*(x) = (1-a)\left(\frac{x}{s}\right)^{1/(n-1)} \qquad \text{for } \ x \in [0, s]$$

where $a = \mu - \mu(1-a)^n$ and $s = n\mu(1-a)^{n-1}$; and $\mathbb{P}(X = 1) = a$. The consumer's payoff from this search can be described as the random variable $Y = \max_j[X_j]$. With probability $1 - (1-a)^n$, Y takes value 1. Moreover, Y has a continuous portion $H(y)$ supported on $[0, s]$:

$$H(y) = \left(F^*\right)^n(x) = (1-a)\left(\frac{x}{s}\right)^{n/(n-1)}$$

The expectation of Y is

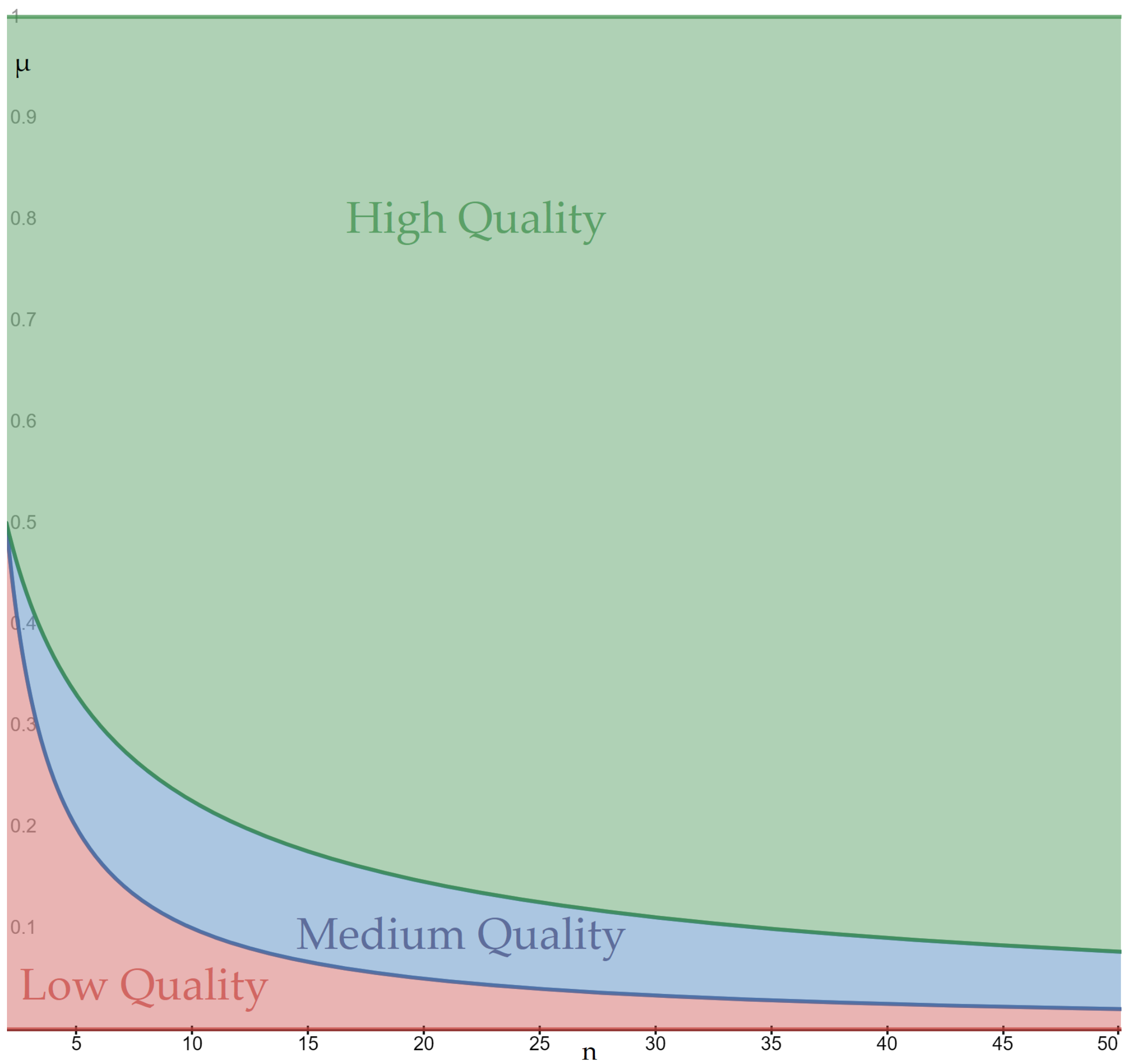

Figure 2: The Three Regions of Average Quality

$$\mathbb{E}[Y] = 1 - (1-a)^n + \int_0^s \frac{n\left(\frac{x}{\mu n}\right)^{\frac{n}{n-1}}}{(n-1)} dx$$

Hence, the expected value to the consumer of Y, $u_{C,L}$, is

$$u_{C,L} = 1 - (1-a)^n + \frac{ns\left(\frac{s}{\mu n}\right)^{\frac{n}{n-1}}}{2n-1} \tag{1}$$

If there are frictions and $\mu \geq 1 - (1/n)^{1/(n-1)}$, then the consumer's expected payoff, $u_{C,F}$, is

$$\begin{aligned} u_{C,F} &= (\mu - c)\sum_{t=0}^{n-1}(1-\mu)^t \\ &= (\mu - c)\frac{1-(1-\mu)^n}{\mu} \end{aligned} \tag{2}$$

The main result:

Theorem 5.1. *Fix a binary prior with mean $\mu > 0$. Then, for any number of firms, n, such that $\mu > \bar{\mu}$, there exists a search cost $\hat{c} > 0$ such that if $c < \hat{c}$, the consumer obtains a higher discounted expected utility from the case with frictions. If n is finite, the frictionless case can lead to strictly higher utility for the consumer.*

Proof. With frictions, the discounted expected utility of the consumer's search is given by Expression 2. As $c \longrightarrow 0$, Expression 2 converges to the expected value of the first order statistic of G_0, i.e. the expected value of the maximum of n draws from distribution G_0:

$$u_{C,F} = 1 - (1-\mu)^n$$

In the frictionless case, each firm will choose the strategy described in Theorem 3.1, which for finite n is $F_i \neq G_0$, $F_i \in M(\mu)$. Recall, from Expression 1, the expected value to the consumer is given by:

$$u_{C,L} = 1 - (1-a)^n + \frac{ns\left(\frac{s}{\mu n}\right)^{\frac{n}{n-1}}}{2n-1}$$

This is strictly increasing in a, and for finite n, $a < \mu$. Thus we must have $u_{C,L} < 1 - (1-\mu)^n = u_{C,F}$. Finally, since Expression 2 is smooth and strictly decreasing in c, the result follows.

■

In Figure 3 we illustrate this dynamic for small n and an expected reward of $\mu = 1/3$. Regardless of how many firms there are, it is welfare improving to the consumer to be slightly impatient and/or have a small search cost. This slight cost leads to the "perfect competition" in information provision.

Because the consumer's search is directed, it becomes important for the firms to choose a signal that entices the consumer to visit them *first*. Crucially, this choice occurs before the consumer commences her search, and this, in tandem with the search cost, forces the firms to compete on information.

Evidently, in the high average quality case, the equilibrium distribution over qualities in the setting with frictions do not converge to to the equilibrium distribution over qualities in the frictionless setting. It is also clear that this is also true in the medium average quality case–γ is independent of c, so with some positive probability (independent of c) the sellers are fully informative (provided $c > 0$), which is not true for $c = 0$. However, there is convergence of distributions over qualities in the low average quality case:

Proposition 5.2. *Let $\mu \leq \underline{\mu}$. Then, as $c \to 0$ the distribution over qualities yielded by the equilibrium strategy of a seller from Proposition 4.6 converges to the distribution over qualities yielded by the equilibrium strategy of a seller from Theorem 3.1.*

6 Robustness

Here we explore two forms of robustness for Proposition 4.1. First, we establish that it holds when the consumer is allowed to select a firm without visiting it as in Doval (2018). Second, we reassure ourselves that an analog to Proposition 4.1 holds for a more general distribution of qualities.

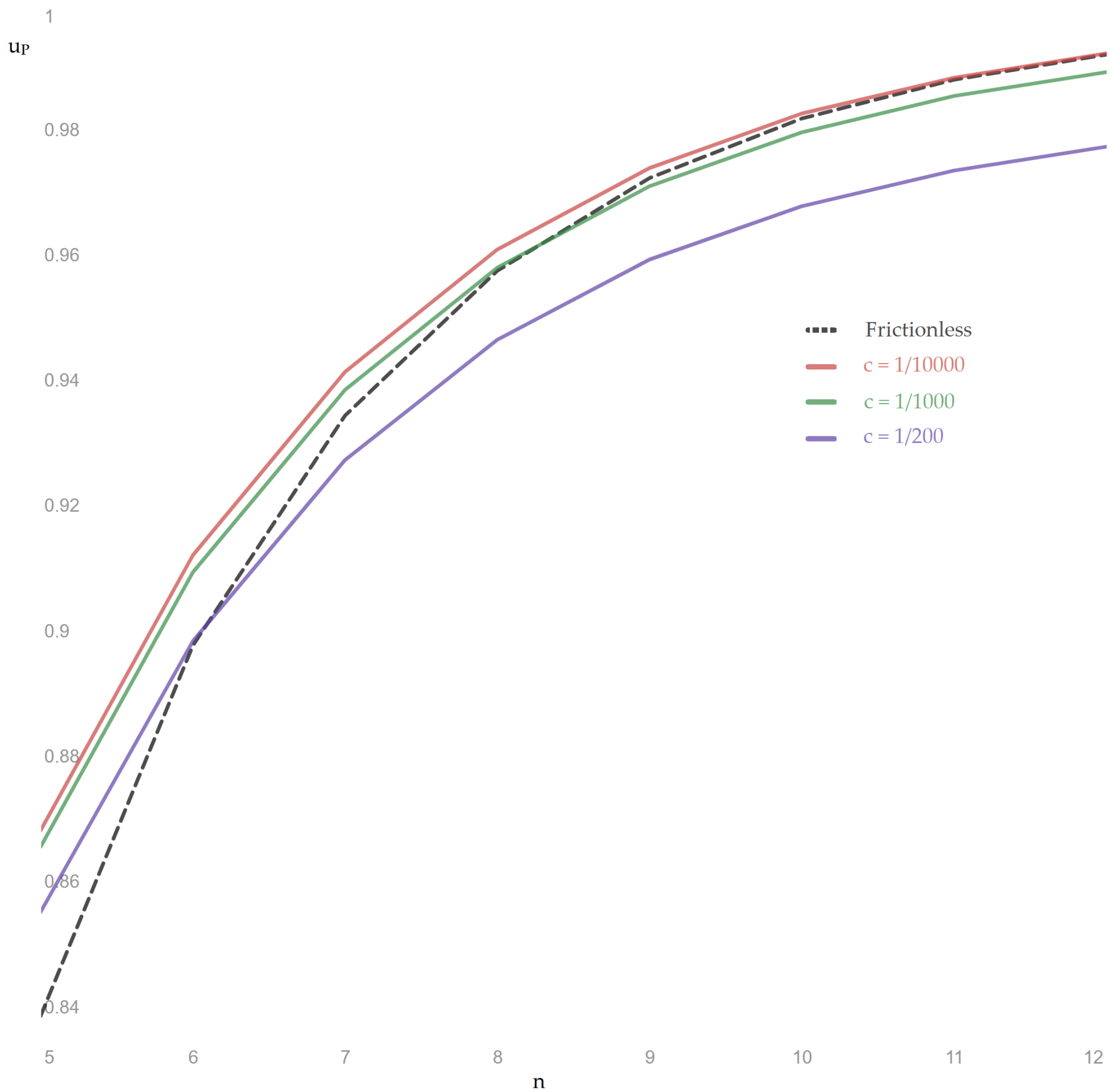

Figure 3: Effect of n on the Consumer's Welfare, for $\mu = 1/3$.

6.1 Whether (or not) to Open Pandora's Box

A recent paper, Doval (2018), explores a modification of the Weitzman search problem in which the consumer need not visit a firm (and pay the inspection cost) to select it. Here we verify that an analog to Proposition 4.1 holds under this alternate formulation. *Viz*, if the product is high average quality then competitions beget full information as the unique symmetric equilibrium in pure strategies:

Proposition 6.1. *Suppose that the consumer may select a firm without inspection. If $\mu \geq \bar{\mu}$ then the unique symmetric pure strategy equilibrium is one in which each firm uses a fully revealing signal.*

Proof. First, suppose that each firm chooses the Bernoulli prior (full information). If a firm deviates then it must deviate to a distribution that is a mean preserving contraction of the distributions chosen by the other firms. Thus, for each pair of firms, the distribution chosen by one firm can be obtained as a mean preserving spread of the distribution chosen by the other. Accordingly, the conditions for Theorem 1 in Doval (2018) hold and so Pandora's Rule is optimal (with a slight modification to the protocol for the final firm). The deviating firm will be visited last, and since $\mu \geq \bar{\mu}$, this deviation is not profitable.

Second, consider any other vector of pure strategies. A firm can deviate profitably by choosing a distribution that is a mean-preserving spread of the on-path distribution (by splitting a small measure from the top of the distribution between small mass points on 0 and 1) and is arbitrarily close to the on-path distribution. Again, Theorem 1 in Doval (2018) applies: thus, the deviating firm will be visited first, securing it a discrete jump in its payoff (as in the proof of Lemma 4.3. ■

Just like Proposition 4.1, Proposition 6.1 guarantees that for any fixed $\mu \in (0, 1)$, there is some N such that if the number of firms, n, is greater than or equal to N, there exists a unique symmetric equilibrium in pure strategies, where each firm chooses a fully informative signal.

6.2 Beyond Two States

What if there are more than two states? The prior distribution over qualities for each firm is distribution G_0 (supported without loss of generality on $[0, 1]$), and now the set of pure strategies for each firm is the set of mean preserving contractions of G_0. Accordingly,

Proposition 6.2. *If*

$$\int_{z_{G_0}}^{1} dG_0(x) \geq \bar{\mu}$$

then a symmetric pure strategy equilibrium exists. Moreover, any such equilibrium must be one in which each firm chooses a distribution that induces the maximal reservation value.

Proof. The proof mimics that of Proposition 4.1. On path, any deviation guarantees that the deviating firm be visited last, which decision's profitability is negated by the assumed condition. Similarly, for any symmetric vector of strategies that does not induce the maximal reservation value, a firm can deviate by being slightly more informative, allowing it to be visited first and selected with probability strictly greater than $1/n$. ∎

Note that this proposition does not imply that the firms provide full information, merely that they choose distributions that yield the highest index value. However, the following corollary provides a sufficient condition for the firms to provide information that yields the consumer a payoff as high as she would receive with full information.

Corollary 6.3. *If*

$$\int_{z_{G_0}}^{1} dG_0(x) \geq \bar{\mu}$$

and the support of G_0 on $[0, z_{G_0}]$ is a singleton then any symmetric pure strategy equilibrium is one in which the consumer obtains her first-best payoff (that which she could receive if each firm was fully informative). There is a symmetric pure strategy equilibrium in which the firms provide full information.

If the support of G_0 on $[0, z_{G_0}]$ is *not* singleton then for those points of support the firms face the same problem as in the frictionless case.

7 Discussion

In this paper we explore the problem of information provision in a sequential, directed search setting. With frictions, the choice of whom to visit, and in what order, becomes paramount to the consumer. Each firm, all else equal, would prefer to be visited first, and this affects its choice of signal. With frictions, as long as the expected value of the quality, μ, is sufficiently high, the unique symmetric equilibrium is one in which each firm chooses a signal that is fully informative. This can be contrasted with the unique symmetric equilibrium in the case without frictions, which, for finite n, is *never* fully informative. The consumer naturally prefers more informative signals, and so as long as the costs are sufficiently low, she counter-intuitively prefers the case with frictions (see Figure 3).

Outside of the high average quality case, search frictions continue to encourage information provision, but are insufficiently powerful to counteract the effect of the low mean: if everyone else provides full information, it is now worthwhile for a firm to provide no information and count on the consumer to visit and select it at the end of an (unsuccessful) search. In the low average quality case, this incentive is strong and firms never provide full information. The medium average quality case (as we might expect) is something of a blend of the other two cases. There the firms occasionally provide full information, but not always.

The structure of the equilibria is notable as well. In each case, there are equilibria in which the consumer is faced with n firms whose binary distributions are all nested within each other. Thus, the information structures are Blackwell comparable. In addition, the optimal search protocol is simple as well: the consumer stops at a firm should the high value realize and continue otherwise. Though this is search in which recall is permitted, the consumer never returns.

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A Appendix (Section 4 Proofs)

A.1 Lemma 4.2 Proof

Proof. Recall,

$$z_{G_0} = \frac{\mu - c}{\mu}$$

Consider any distribution $F_i \neq G_0$, $F_i \in M(\mu)$ such that $\int_{z_{F_i}}^1 x dF_i(x) = \mu$. Since $F_i \neq G_0$, we must have $F(z_{F_i}) = k(1-\mu) < (1-\mu)$, for some $1 > k \geq 0$. Then,

$$z_{F_i} = \frac{\mu - c}{1 - k(1-\mu)} \tag{A1}$$

Expression $A1$ is strictly increasing in k and thus

$$z_{F_i} = \frac{\mu - c}{1 - k(1-\mu)} < \frac{\mu - c}{1 - (1-\mu)} = z_{G_0}$$

Next, consider any distribution $H_i \neq G_0$, $H_i \in M(\mu)$ such that $\int_{z_{H_i}}^1 x dH_i(x) = t\mu$, for some $0 < t < 1$. Since $H_i \neq G_0$, we must have $1 - t\mu \geq F(z_{H_i}) > 0$. We have

$$z_{H_i} = \frac{t\mu - c}{1 - F(z_{H_i})} \tag{A2}$$

Expression $A2$ is strictly increasing in $F(z_{H_i})$, which is bounded above by $1 - t\mu$, and thus

$$z_{H_i} = \frac{t\mu - c}{1 - F(z_{H_i})} \leq \frac{t\mu - c}{1 - (1 - t\mu)} < \frac{\mu - c}{1 - (1 - \mu)} = z_{G_0}$$

where we used the fact that $(t\mu - c) / (1 - (1 - t\mu))$ is strictly increasing in t.

■

A.2 Lemma 4.3 Proof

Proof. Suppose there is a symmetric equilibrium in which each firm chooses a distribution, H_i that induces some $z_{H_i} < z_{G_0}$. Suppose that firm 1 deviates and instead chooses distribution $F_i \in M(\mu)$ ($F_i \neq G_0$, and $F_i \neq H_i$), such that $\int_{z_{F_i}}^{1} x dF_i(x) = \mu$. In other words, the mass of this distribution below its reservation value consists entirely of a point mass on 0. Since $F_i \neq G_0$, we must have $F(z_{F_i}) = k(1 - \mu) < (1 - \mu)$, for some $1 > k \geq 0$. Then,

$$z_{F_i} = \frac{\mu - c}{1 - k(1 - \mu)} \tag{A3}$$

The right-hand side of Equation $A3$ is strictly and smoothly increasing in k and thus there is some $\hat{k}$ such that for all $k > \hat{k}$, $z_{F_i} > z_{H_i}$. Moreover, under this distribution it is clear that $\mathbb{P}(X_i > z_{F_i}) \geq \mathbb{P}(X_i > z_{H_i}) > 1/n$. Thus, firm 1 has achieved a profitable deviation. Note that for any $1 > k \geq 0$ it is easy to construct a corresponding F_i.

If each firm chooses G_0 then there is no profitable deviation, since any deviation will induce a strictly lower z. That firm will be visited last, and its payoff will be at most $(1-\mu)^{n-1} \leq 1/n$, by assumption. ■

A.3 Proposition 4.4 Proof

Proof. First, a lemma:

Lemma A.1. *There is no pure strategy equilibrium in which firms play distributions with any* $z_{F_i} < z_{G_0}$.

Proof. Suppose for the sake of contradiction that there is such an equilibrium. Each firm plays strategy $F_i \in M(\mu)$, $F_i \neq G_0$. For convenience we may drop the subscript and define distribution $H := F^{n-1}(x)$. Firm i's payoff is given by

$$u_i = \frac{1}{n}\left[1 - F(z_F) + \int_0^{z_F} H(x)dF(x)\right] + \frac{1}{n}\left[F(z_F)(1 - F(z_F)) + \int_0^{z_F} H(x)dF(x)\right]$$
$$+ \cdots + \frac{1}{n}\left[F(z_F)^{n-1}(1 - F(z_F)) + \int_0^{z_F} H(x)dF(x)\right]$$

Or,

$$u_i = \frac{1}{n}(1 - F(z_F))\sum_{t=0}^{n-1} F(z_F)^t + \int_0^{z_F} H(x)dF(x)$$

Suppose that firm j deviates by increasing the weight placed on 1 by ϵ, removing weight $\gamma = \int_0^{\eta} dF(x)$, $\eta < z_F$; and increasing the weight placed on 0 by ρ, with $\epsilon + \rho = \gamma$ and $\epsilon = \int_0^{\eta} x dF(x)$.[12] Call this strategy $\hat{F}$. Through similar logic to that used in Lemma 4.2, *supra*, $z_{\hat{F}} > z_F$. Firm i's expected payoff from this deviation, $\hat{u}_i$, is

$$\hat{u}_i = 1 - \hat{F}(z_F) + \int_0^{z_F} H(x)d\hat{F}(x)$$

Then, as $\eta, \epsilon, \rho \longrightarrow 0$, $\hat{u}_i - u_i$ converges to

$$(1 - F(z_F))\left(1 - \frac{1}{n}\sum_{t=0}^{n-1} F(z_F)^t\right) > 0$$

Thus, firm i has a profitable deviation by choosing sufficiently small η, ϵ and ρ. ■

From Lemma 4.2, we see that the uniquely maximal reservation value assigned to a firm is that induced by a fully informative signal. Furthermore, Lemma A.1 dictates that the symmetric pure strategy equilibrium must consist of distributions that induce the maximal reservation value. Thus we conclude that the symmetric pure strategy equilibrium is one where

[12] This is always possible unless all of the weight of F is above z_F. However, if all of the weight of F were above z_F then z_F would be minimized, and there is obviously no equilibrium in which each firm chooses a distribution that induces the minimal z.

each firm chooses the prior distribution G_0. Then, we see that there is also a profitable deviation from this strategy profile. Since $(1-\mu)^{n-1} > 1/n$, a deviation to a completely uninformative strategy, μ with probability 1, obtains firm i an expected payoff of $(1-\mu)^{n-1} > 1/n$ and we conclude that there is no symmetric equilibrium in pure strategies.

■

A.4 Proposition 4.5 Proof

Proof. First we note the following remark:

Remark A.2. Any pure strategy is equivalent to a strategy in which the portion of the distribution above z_{F_i} consists of a single point mass on some point $m_i > z_{F_i}$.

Thus, we may restrict the set of pure strategies (and set of support for the mixed strategies) to those distributions with a single point mass above the z induced by the distribution. That is, without loss of generality, a firm's pure strategy has a point mass of weight p_i on some $m_i > z_{F_i}$.

We may now prove the proposition through contradiction. Suppose that there is a symmetric mixed strategy equilibrium in which there is positive measure induced on some $\hat{z} < z_{G_0}$. That is, with probability $q > 0$ a firm chooses a distribution with reservation value $\hat{z}$. Then, with probability $q^n > 0$ each firm chooses a distribution with the same reservation value. Define $a_i := \mathbb{E}_\Gamma[p_i | z = \hat{z}]$ (where Γ is the mixed strategy distribution over pure strategies) and note that since we are considering symmetric equilibria we may drop the subscript. Firm i's payoff in this proposed equilibrium is

$$\begin{aligned} u_i &= q^n \left[\frac{1}{n}(a+\zeta) + \frac{1}{n}(a(1-a)+\zeta) + \cdots + \frac{1}{n}(a(1-a)^{n-1}+\zeta)\right] + \Psi + \Upsilon \\ &= q^n \left[\frac{1}{n}a + \frac{1}{n}\left(a(1-a)\right) + \cdots + \frac{1}{n}\left(a(1-a)^{n-1}\right) + \zeta\right] + \Psi + \Upsilon \end{aligned}$$

where ζ is firm i's expected payoff when no firm obtains a realization greater than $\hat{z}$, given that each realization of the mixed strategy chosen by each firm is some distribution that induces $\hat{z}$, Ψ is the sum of firm i's expected payoffs in the instances where only $1, 2, \ldots, n-2$ of the other firms' mixed strategy realizations is/are some distribution that induces $\hat{z}$, and Υ

is firm i's expected payoff when no firm's mixed strategy realization is any distribution that induces $\hat{z}$.

Now, suppose that firm i deviates and instead chooses a mixed strategy that has a positive measure q induced on some $\hat{z} + \epsilon$, where $\hat{z} < \hat{z} + \epsilon < z_{G_0}$ and is otherwise identical to the original mixed strategy. Moreover, suppose that each distribution in the support of the mixed strategy that induces $\hat{z} + \epsilon$ is a distribution obtained by adding weight η to m and ρ to 0 and excising some portion of the nonzero portion of the distribution (below $\hat{z}$)[13] from a distribution in the support of the mixed strategy that induces $\hat{z}$ in the original proposed equilibrium.

As $\eta \longrightarrow 0$ (and thus $\epsilon \longrightarrow 0$), firm i's payoff will converge to $u_i' = q^n(a + \zeta) + \Psi' + \Upsilon > u_i$ since $a > a/n + (1/n)\left(a(1-a)\right) + \cdots + (1/n)\left(a(1-a)^{n-1}\right)$ and analogously $\Psi' > \Psi$. We have established a profitable deviation. ∎

B Propositions 4.6 and 4.7 Proof

Recall
$$\underline{\mu} := \frac{1}{n}, \quad \text{and} \quad \bar{\mu} := 1 - \left(\frac{1}{n}\right)^{\frac{1}{n-1}}$$

It suffices to prove Proposition 4.7 since the proof for Proposition 4.6 follows analogously, with the point mass γ equal to 0.

Let's look for an equilibrium in which each firm chooses a distribution over reservation values with a point mass of size $\gamma \geq 0$ on z_{G_0} and a continuous portion Φ on $[\underline{z}, \bar{z}]$ where
$$\Phi(\bar{z}) = 1 - \gamma, \quad \text{and} \quad \int_{\underline{z}}^{\bar{z}} w\phi(w)dw = \mu - \gamma$$

Moreover, we assume that each z is induced by a binary distribution $a(z) > z$ with probability $p(z)$ and $b(z) < z$ with $1 - p(z)$, where a is strictly increasing in z and b is strictly decreasing in z.

If firm 1 plays z_{G_0} then her expected payoff, when played against the equilibrium mixed

[13] Done in such a way that the resulting distribution induces $\hat{z} + \epsilon$.

strategies of the others, is

$$u_1 = \frac{((1-\mu)\,\gamma)^n}{n\gamma}$$
$$+\mu\left[(1-\gamma)^{n-1} + \binom{n-1}{1}(1-\gamma)^{n-2}\gamma\left(\frac{\mu}{2}+1-\mu\right) + \cdots + \gamma^{n-1}\sum_{i=0}^{n-1}\frac{1}{n-i}\binom{n-1}{i}\mu^{n-1-i}(1-\mu)^i\right]$$

which, after appealing to the Binomial theorem, reduces to

$$u_1 = \frac{1 + ((1-\mu)\,\gamma)^n - (1-\mu\gamma)^n}{n\gamma}$$

This must equal $1/n$, so

$$1-\gamma = (1-\mu\gamma)^n - ((1-\mu)\,\gamma)^n$$

Next, define the polynomial $f(\gamma;\mu,n) = f(\gamma)$ as

$$f(\gamma;\mu,n) := (1-\mu\gamma)^n - ((1-\mu)\,\gamma)^n - (1-\gamma)$$

Claim B.1. *In any symmetric equilibrium, $\gamma = 0$ if and only if $\mu \leq \underline{\mu}$; $\gamma \in (0,1)$ if and only if $\bar{\mu} > \mu > \underline{\mu}$; and $\gamma = 1$ if and only if $\mu \geq \bar{\mu}$.*

Proof. Evidently, f has roots at both $\gamma = 0$ and $\gamma = 1$. Moreover,

$$f'(\gamma) := -n\mu\,(1-\mu\gamma)^{n-1} - n\,(1-\mu)\,((1-\mu)\,\gamma)^{n-1} + 1$$

$f'(0) = 1 - n\mu$, which is (strictly) positive if and only if $\mu(<) \leq \underline{\mu}$. Moreover, $f'(1) = 1 - n\,(1-\mu)^{\frac{1}{n-1}}$, which is (strictly) positive if and only if $\mu(>) \geq \bar{\mu}$. Next,

$$f''(\gamma) := n(n-1)\mu^2\,(1-\mu\gamma)^{n-2} - n(n-1)\,(1-\mu)^2\,((1-\mu)\,\gamma)^{n-2}$$

which is positive for all $0 \leq \gamma \leq \hat{\gamma}$ and negative for all $1 \geq \gamma \geq \hat{\gamma}$. Finally, we conclude that f has a root in the interval $(0,1)$ if and only if $\bar{\mu} > \mu > \underline{\mu}$. ■

By the implicit function theorem we see that γ is continuous in μ. Moreover, f is strictly decreasing in μ. Hence, γ must be strictly increasing in μ (on $(\underline{\mu}, \bar{\mu})$).

Next, let's characterize the continuous portion of the distribution. Let firm 1 deviate to distribution 0 with $1-q$ and v with probability q. Observe that

$$q = \frac{\mu - c}{z}, \quad \text{and} \quad 1 - q = \frac{z - (\mu - c)}{z}$$

firm 1's payoff is

$$u_1 = q\left(\Phi(z) + \int_z^{\bar{z}} (1 - p(w))\,\phi(w)dw + \gamma(1-\mu)\right)^{n-1} + (1-q)\,(\gamma(1-\mu))^{n-1} \qquad (A4)$$

or,

$$u_1 = q\left(\Phi(z) + \int_z^{\bar{z}} (1 - p(w))\,\phi(w)dw + \gamma(1-\mu)\right)^{n-1} + \left(\frac{z-(\mu-c)}{z}\right)(\gamma(1-\mu))^{n-1}$$

which must equal $1/n$. Thus,

$$q\left(\Phi(z) + \int_z^{\bar{z}} (1 - p(w))\,\phi(w)dw + \gamma(1-\mu)\right)^{n-1} = \frac{1}{n} - \left(\frac{z-(\mu-c)}{z}\right)(\gamma(1-\mu))^{n-1}$$

or,

$$\begin{aligned}\left(\Phi(z) + \int_z^{\bar{z}} (1 - p(w))\,\phi(w)dw + \gamma(1-\mu)\right)^{n-1} &= \frac{z}{n\,(\mu-c)} - \left(\frac{z}{\mu-c} - 1\right)(\gamma(1-\mu))^{n-1} \\ &= \frac{z\,(1-n\beta) + n\,(\mu-c)\,\beta}{n\,(\mu-c)} =: \frac{\xi(z)}{n\,(\mu-c)}\end{aligned} \qquad (A5)$$

where

$$\beta := (\gamma(1-\mu))^{n-1}$$

Then,

$$\int_z^{\bar{z}} (1 - p(w))\,\phi(w)dw + \gamma(1-\mu) = \left(\frac{\xi(z)}{n\,(\mu-c)}\right)^{\frac{1}{n-1}} - \Phi(z) \qquad (A6)$$

Using Leibniz's rule,

$$p(z)\phi(z) = \left(\frac{1-n\beta}{n-1}\right)\left(\frac{1}{n\,(\mu-c)}\right)^{\frac{1}{n-1}} \xi\,(z)^{\frac{1}{n-1}-1} \qquad (A7)$$

Next,

Claim B.2. *If an equilibrium of the proposed form exists, the lower bound* $\bar{b} := b\,(\bar{z})$ *of the bs must be equal to* 0.

Proof. Suppose for the sake of contradiction that $\bar{b} > 0$. Let firm 1 choose $\bar{z}$ corresponding to distribution $\bar{a} > \bar{z}$ with q and $\bar{b}$ with $1-q$. By construction,

$$q = \frac{\mu - c - \bar{b}}{\bar{z} - \bar{b}}$$

which is strictly decreasing in $\bar{b}$. Then, his payoff is

$$\begin{aligned} u_1 &= q\left(1-\gamma+\gamma(1-\mu)\right)^{n-1}+(1-q)\left(\gamma(1-\mu)\right)^{n-1} \\ &= q\left(1-\gamma\mu\right)^{n-1}+(1-q)\left(\gamma(1-\mu)\right)^{n-1} \end{aligned}$$

which is evidently strictly increasing in q and hence strictly decreasing in $\bar{b}$. Thus, a firm has a profitable deviation by choosing a lower $b(\bar{z})$. This can be done so long as $\bar{b} > 0$. ■

Consequently, $b\left(\bar{z}\right)$ must equal 0. Hence,

$$\bar{q} = \frac{\mu - c}{\bar{z}}$$

and substituting this into Equation $A4$, evaluating that equation at $\bar{z}$, and setting it equal to $1/n$, we have

$$\left(\frac{\mu - c}{\bar{z}}\right)(1-\gamma\mu)^{n-1}+\left(1-\frac{\mu - c}{\bar{z}}\right)\left(\gamma(1-\mu)\right)^{n-1} = \frac{1}{n}$$

which, rearranged, is

$$\bar{z} = \frac{n\left(\mu - c\right)\left(\alpha - \beta\right)}{1 - n\beta} \tag{A8}$$

where

$$\alpha := (1-\gamma\mu)^{n-1}$$

Note that if $\gamma = 0$, $\bar{z} = n\left(\mu - c\right)$, and if $\gamma = 1$, $\bar{z} = 0$. Next, on path, firm 1's payoff is

$$\begin{aligned} u_1 =& p(z)\left(\Phi(z)+\int_z^{\bar{z}}\left(1-p(w)\right)\phi(w)dw+\gamma(1-\mu)\right)^{n-1} \\ &+\left(1-p(z)\right)\left(\int_z^{\bar{z}}\left(1-p(w)\right)\phi(w)dw+\gamma(1-\mu)\right)^{n-1} \end{aligned}$$

This must equal $1/n$ for any (on-path) pure strategy. Let's substitute Equations $A5$ and $A6$ into this:

$$p(z)\frac{\xi(z)}{n\left(\mu - c\right)}+\left(1-p(z)\right)\left(\left(\frac{\xi(z)}{n\left(\mu - c\right)}\right)^{\frac{1}{n-1}}-\Phi(z)\right)^{n-1} = \frac{1}{n}$$

Rearranging,

$$\begin{aligned} \Phi(z) &= \left(\frac{\xi(z)}{n\left(\mu - c\right)}\right)^{\frac{1}{n-1}}-\left(\frac{(\mu - c)-\xi(z)p(z)}{n\left(\mu - c\right)\left(1-p(z)\right)}\right)^{\frac{1}{n-1}} \\ &= \left(\frac{1}{n\left(\mu - c\right)}\right)^{\frac{1}{n-1}}\left[\xi(z)^{\frac{1}{n-1}}-\left(\frac{(\mu - c)-\xi(z)p(z)}{1-p(z)}\right)^{\frac{1}{n-1}}\right] \end{aligned}$$

Using $p(z)z = \mu - c - b(z)(1 - p(z))$ and the definition of ξ, this becomes

$$\Phi(z) = \left(\frac{1}{n(\mu - c)}\right)^{\frac{1}{n-1}} \left\{\xi(z)^{\frac{1}{n-1}} - [b(z)(1 - n\beta) + n(\mu - c)\beta]^{\frac{1}{n-1}}\right\} \tag{A9}$$

Differentiating this expression, we have

$$\phi(z) = \left(\frac{1 - n\beta}{n - 1}\right)\left(\frac{1}{n(\mu - c)}\right)^{\frac{1}{n-1}} \left\{\xi(z)^{\frac{1}{n-1}-1} - [b(z)(1 - n\beta) + n(\mu - c)\beta]^{\frac{1}{n-1}-1} b'(z)\right\}$$

Then, combining this with Equation $A7$, we have

$$\xi(z)^{\frac{1}{n-1}-1} = p(z)\left\{\xi(z)^{\frac{1}{n-1}-1} - [b(z)(1 - n\beta) + n(\mu - c)\beta]^{\frac{1}{n-1}-1} b'(z)\right\} \tag{A10}$$

Moreover, since

$$p(z) = \frac{\mu - c - b(z)}{z - b(z)}$$

$$\xi(z)^{\frac{1}{n-1}-1}(z - (\mu - c)) = (b(z) - (\mu - c))[b(z)(1 - n\beta) + n(\mu - c)\beta]^{\frac{1}{n-1}-1} b'(z)$$

Solving this differential equation, we have

$$\begin{aligned} &((1 - n\beta)b(z) + n(\mu - c)\beta)^{\frac{1}{n-1}}((1 - n\beta)b(z) - (1 - \beta)n(\mu - c)) \\ &= ((1 - n\beta)z + n(\mu - c)\beta)^{\frac{1}{n-1}}((1 - n\beta)z - (1 - \beta)n(\mu - c)) \end{aligned} \tag{A11}$$

where we used the fact that $b(\bar{z}) = 0$ and $b(\mu - c) = \mu - c$. If $\gamma = 0$ then this simplifies to

$$b^{\frac{n}{n-1}} - \bar{z}b^{\frac{1}{n-1}} = z^{\frac{n}{n-1}} - \bar{z}z^{\frac{1}{n-1}}$$

Next, we can verify Proposition 4.7 directly. Recall that it states that there is a symmetric equilibrium given by one constant, γ; and four functions, $a(z)$, $b(z)$, $p(z)$ and $\Phi(z)$; where $b(z)$ and $\Phi(z)$ are given in Equations $A11$ and $A9$, respectively.

Proof. On path each firm gets $1/n$. Suppose firm 1 deviates to choose some distribution $t > z_1$ with probability q and distribution H on $[0, z_1]$, where $\int_0^{z_1} x\,dH(x) = v$, $v + tq = \mu$, and $z_1 = t - c/q$.

Case 1: $z_1 < \bar{z}$. Without loss of generality, let H be supported on $[0, \mu - c]$. Then, firm 1's payoff from deviating is

$$\begin{aligned} u_1 = &q\left(\Phi(z) + \int_{z_1}^{\bar{z}}(1 - p(w))\phi(w)dw + \gamma(1 - \mu)\right)^{n-1} \\ &+ \int_0^{\mu - c}\left(\int_{z(x)}^{\bar{z}}(1 - p(w))\phi(w)dw + \gamma(1 - \mu)\right)^{n-1} dH(x) \end{aligned}$$

where $z(\cdot) := b^{-1}(\cdot)$. Using Expression $A5$, this becomes

$$u_1 = q\frac{\xi(z)}{n(\mu - c)} + \int_0^{\mu-c} \left(\int_{z(x)}^{\bar{z}} (1 - p(w))\, \phi(w) dw + \gamma(1-\mu)\right)^{n-1} dH(x)$$

which becomes

$$u_1 = q\frac{\xi(z)}{n(\mu - c)} + \int_0^{\mu-c} \frac{b(z(x))\,(1 - n\beta) + n(\mu - c)\,\beta}{n(\mu - c)} dH(x)$$

This becomes

$$u_1 = \frac{qz}{n(\mu - c)}(1 - n\beta) + q\beta + \frac{v}{n(\mu - c)}(1 - n\beta) + (1-q)\beta \tag{A12}$$

since $b(z(x)) = b\left(b^{-1}(x)\right) = x$, and where we used the definition of ξ. Equation $A12$ reduces to $1/n$, as required.

Case 2: $z_1 \geq \bar{z}$. Again, let H be supported on $[0, \mu - c]$. Now firm 1 is visited first, unless someone else provides full information, so she gets

$$u_1 = q(1 - \gamma\mu)^{n-1} + \int_0^{\mu-c} \left(\int_{z(x)}^{\bar{z}} (1 - p(w))\, \phi(w) dw + \gamma(1-\mu)\right)^{n-1} dH(x)$$

Or,

$$u_1 = q\alpha + (1-q)\beta + \frac{v}{n(\mu - c)}(1 - n\beta)$$

Using $v = \mu - c - qz_1$

$$u_1 = q\alpha + (1-q)\beta + \frac{\mu - c - qz_1}{n(\mu - c)}(1 - n\beta)$$

Finally, since $z_1 \geq \bar{z}$ we have (using Equation $A8$)

$$\begin{aligned} q\alpha + (1-q)\beta + \frac{\mu - c - qz_1}{n(\mu - c)}(1 - n\beta) &\leq q\alpha + (1-q)\beta + \frac{\mu - c - q\frac{n(\mu-c)(\alpha-\beta)}{1-n\beta}}{n(\mu - c)}(1 - n\beta) \\ &\leq q\alpha + (1-q)\beta + \frac{1}{n} - \beta - q\alpha + q\beta = \frac{1}{n} \end{aligned}$$

and so there is no profitable deviation. ∎

Next, we verify the monotonicity of a, b, and p.

Claim B.3. *b and p are decreasing in z, and a is increasing in z.*

Proof. Using Equation $A11$, we take the derivative of b with respect to z implicitly, which yields (omitting the arguments of b and b')

$$b' = -\left(\frac{r(b)}{t(z)}\right)^{\frac{n-2}{n-1}}\left(\frac{z-(\mu-c)}{(\mu-c)-b}\right) \tag{A13}$$

where

$$r(b) := (1-n\beta)\, b(z) + n(\mu-c)\beta, \quad \text{and} \quad t(z) := (1-n\beta)\, z + n(\mu-c)\beta$$

Since $z \geq (\mu-c) \geq b$, this expression is negative. The proof with regard to a is a little more involved. Recall,

$$p(z) = \frac{\mu-c-b(z)}{z-b(z)}$$

Then, by the chain rule

$$p' = \frac{-(z-(\mu-c))\, b' - ((\mu-c)-b)}{(z-b)^2}$$

Substituting in using b', we have

$$p' = \frac{\left(\frac{r(b)}{t(z)}\right)^{\frac{n-2}{n-1}}(z-(\mu-c))^2 - ((\mu-c)-b)^2}{((\mu-c)-b)\,(z-b)^2}$$

Furthermore, since

$$a(p) = z + \frac{c}{p(z)}$$

by the chain rule

$$a' = 1 - \frac{c}{p^2}p'$$

Consequently it suffices to show that p' is negative, which holds if and only if

$$(r(b))^{\frac{n-2}{n-1}}(z-(\mu-c))^2 - (t(z))^{\frac{n-2}{n-1}}((\mu-c)-b(z))^2 \leq 0$$

Define function $v\,(z, b\,(z))$ as

$$v\,(z, b\,(z)) := r^{\frac{n-2}{n-1}}(z-(\mu-c))^2 - t^{\frac{n-2}{n-1}}((\mu-c)-b)^2$$

where we omitted arguments. Then,

$$\begin{aligned}\frac{d}{dz}v\,(z, b\,(z)) =& 2\,(\mu-c-b)\, t^{\frac{n-2}{n-1}} b' + 2\,(z-(\mu-c))\, r^{\frac{n-2}{n-1}} \\ &+ \frac{n-2}{n-1}(1-n\beta)\,(z-(\mu-c))^2\, r^{\frac{n-2}{n-1}-1} b' - \frac{n-2}{n-1}(1-n\beta)\,(\mu-c-b)^2\, t^{\frac{n-2}{n-1}-1}\end{aligned}$$

Substituting in for the first b', we have

$$\frac{d}{dz}v(z, b(z)) = -2(z-(\mu-c))\, r^{\frac{n-2}{n-1}} + 2(z-(\mu-c))\, r^{\frac{n-2}{n-1}} + \frac{n-2}{n-1}(1-n\beta)(z-(\mu-c))^2\, r^{\frac{n-2}{n-1}-1} b' - \frac{n-2}{n-1}(1-n\beta)(\mu-c-b)^2\, t^{\frac{n-2}{n-1}-1}$$

which reduces to

$$\frac{d}{dz}v(z, b(z)) = \frac{n-2}{n-1}(1-n\beta)(z-(\mu-c))^2\, r^{\frac{n-2}{n-1}-1} b' - \frac{n-2}{n-1}(1-n\beta)(\mu-c-b)^2\, t^{\frac{n-2}{n-1}-1}$$

which is negative since b' is negative. Thus v is decreasing in z. Finally,

$$v((\mu-c), b(\mu-c)) = v((\mu-c), (\mu-c)) = 0$$

Thus p must be (at least weakly) decreasing in z so a is increasing in z. ■

To wrap things up, we determine $p(\underline{z})$.

Claim B.4. $p(\underline{z}) = 1/2$

Proof. Using Equation $A13$, we have

$$\begin{aligned}\lim_{z\to\underline{z}} b'(z) &= \lim_{z\to\underline{z}}\left(-\left(\frac{r(b)}{t(z)}\right)^{\frac{n-2}{n-1}}\left(\frac{z-(\mu-c)}{(\mu-c)-b}\right)\right)\\ &= \lim_{z\to\underline{z}}\left(-\left(\frac{r(b)}{t(z)}\right)^{\frac{n-2}{n-1}}\right)\lim_{z\to\underline{z}}\left(\frac{z-(\mu-c)}{(\mu-c)-b}\right)\end{aligned}$$

which simplifies to

$$\lim_{z\to\underline{z}} b'(z) = -\lim_{z\to\underline{z}}\left(\frac{z-(\mu-c)}{(\mu-c)-b}\right) \tag{A14}$$

Define α as

$$\alpha := \lim_{z\to\underline{z}} b'(z)$$

Then, by L'Hôpital's rule, Equation $A14$ reduces to

$$\alpha = \frac{1}{\alpha}$$

or $\alpha = -1$. Hence, $b'(\underline{z}) = -1$. Finally, we can use this to evaluate Equation $A10$ at $z = \underline{z}$, which reduces to

$$\xi(\underline{z})^{\frac{1}{n-1}-1} = 2p(\underline{z})\,\xi(\underline{z})^{\frac{1}{n-1}-1}$$

whence we conclude that $p(\underline{z}) = 1/2$. ■

C Proposition 5.2 Proof

Proof. As $c \longrightarrow 0$, the equilibrium strategies described in Proposition 4.6 will converge to strategies in which $b(z)$ takes values in $[0, \mu]$ and $a(z)$ takes values in $[\mu, n\mu]$. Accordingly, for values $x \in [0, \mu]$, the pseudo-cdf[14] of values, will be given by

$$\int_{z(x)}^{\bar{z}} (1 - p(w))\, \phi(w) dw$$

where $z(x) = b^{-1}(x)$. Using the fact that

$$\int_{z}^{\bar{z}} (1 - p(w))\, \phi(w) dw = \left(\frac{z}{n(\mu - c)} \right)^{\frac{1}{n-1}} - \Phi(z)$$

and

$$\Phi(z) = \left(\frac{1}{n\,(\mu - c)} \right)^{\frac{1}{n-1}} \left\{ z^{\frac{1}{n-1}} - b(z)^{\frac{1}{n-1}} \right\}$$

we have (for $c = 0$)

$$\int_{z(x)}^{\mu} (1 - p(w))\, \phi(w) dw = \left(\frac{(z(x))}{n\mu} \right)^{\frac{1}{n-1}} - \left(\frac{1}{n\mu} \right)^{\frac{1}{n-1}} \left\{ (z(x))^{\frac{1}{n-1}} - b\,(z(x))^{\frac{1}{n-1}} \right\} = \left(\frac{x}{n\mu} \right)^{\frac{1}{n-1}}$$

with associated density

$$\left(\frac{1}{n-1} \right) \left(\frac{1}{n\mu} \right)^{\frac{1}{n-1}} x^{\frac{1}{n-1} - 1} \tag{3}$$

For values $x \in [\mu, n\mu]$, the pdf of values will be given by $p\,(\tilde{z}(x))\, \phi\,(\tilde{z}(x))$ where $\tilde{z}(x) = a^{-1}(x)$. Since

$$p(z)\phi(z) = \left(\frac{1}{n-1} \right) \left(\frac{1}{n\,(\mu - c)} \right)^{\frac{1}{n-1}} z^{\frac{1}{n-1} - 1}$$

and using the fact that when $c \longrightarrow 0$, $a \longrightarrow z$, we have (for $c = 0$),

$$p\,(\tilde{z}(x))\, \phi\,(\tilde{z}(x)) = \left(\frac{1}{n-1} \right) \left(\frac{1}{n\mu} \right)^{\frac{1}{n-1}} x^{\frac{1}{n-1} - 1}$$

Using Expression 3, we conclude the distribution over qualities begotten by the mixed strategy of a seller converges to a distribution with density

$$\left(\frac{1}{n-1} \right) \left(\frac{1}{n\mu} \right)^{\frac{1}{n-1}} x^{\frac{1}{n-1} - 1}$$

supported on $[0, n\mu]$, which corresponds precisely to the distribution over qualities described in Theorem 3.1. ∎

[14] This satisfies all of the properties of a cdf save one: it does not sum to 1 on $[0, \mu]$.