Persuading a Consumer to Visit
Online Appendix

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1 Theorem 3.1 and Corollary 3.2 Proof

The purpose of this supplement is to prove Theorem 3.1 and Corollary 3.2 from Whitmeyer (2018). Note that these results, or variants thereof, were derived independently in several other papers–Spiegler (2006), who solved an problem isomorphic to this one for the mean fixed at 1/2; Boleslavsky and Cotton (2015) and Albrecht (2017), who solved a possibly asymmetric two player version; and Au and Kawai (2020).

This appendix is a condensed and much abridged version of the (now defunct) working paper Hulko and Whitmeyer (2017).

Theorem 1.1 (Theorem 3.1 in Whitmeyer (2018))**.** *Let the search cost* $c = 0$*.*

1. *If* $\mu \geq \underline{\mu}$ *then the unique symmetric pure strategy equilibrium is for each firm to play distribution* F^**, defined as*

$$F^*(x) = \begin{cases} (1-a)\left(\frac{x}{s}\right)^{\frac{1}{n-1}}, & 0 \leq x < s \\ (1-a), & s \leq x < 1 \\ 1, & 1 = x \end{cases}$$

where

$$a = \mu - \mu(1-a)^n, \quad \textit{and} \quad s = n\mu(1-a)^{n-1}$$

2. *If* $\mu \leq \underline{\mu}$ *then the unique symmetric pure strategy equilibrium is for each firm to play distribution* F^**, defined as*

$$F^* = \left(\frac{x}{n\mu}\right)^{\frac{1}{n-1}}, \quad 0 \leq x \leq n\mu$$

and

Corollary 1.1 (Corollary 3.2 in Whitmeyer (2018))**.** *Let* $\mu > \underline{\mu}$*. If the number of firms is increased, the weight placed on* 1 *in the symmetric equilibrium must increase. That is,* a *is strictly increasing in* n*.*

In the limit, as the number of firms, n*, becomes infinitely large, the weight on* 1 *converges to* μ*. That is, the equilibrium distribution converges to a distribution with support on two points,* 1 *and* 0*.*

We begin with the following lemma

Lemma 1.1. ***(Frictionless)*** *There are no symmetric equilibria with point masses on any point in the interval* $[0,1)$.

Proof. First, we establish that there are no symmetric Nash Equilibria where firms choose discrete distributions supported on $N(<\infty)$ points.

It is easy to see that there is no symmetric equilibrium in which each firm chooses a distribution consisting of a single point mass. Such a distribution could only consist of distribution with weight 1 placed on μ and would yield to each firm a payoff of $1/n$. However, there is a profitable deviation for a firm to instead place weight $1-\epsilon$ on $\mu+\eta$ and weight ϵ on 0 ($\epsilon, \eta > 0$). In doing so, this firm could achieve a payoff arbitrarily close to 1.

Now, assume $\infty > N \geq 2$. Observe that a strategy consists of a choice of probabilities $\{p_1, p_2, \dots, p_N\}$, $p_i \in [0,1]\ \forall i$, $\sum_{i=1}^N p_i = 1$ and support $a_1 < a_2 < \dots < a_N \in [0,1]$ such that $\sum_{i=1}^N a_i p_i = \mu$.

The expected payoff to each firm from playing an arbitrary strategy, $S_1 = S_2 = \dots = S_n = E$, is

$$u_i(S_i, S_{-i}) = \sum_{j=0}^{N-1}\left(\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{1}{m-i}p_{N-j}^{n-i}\left(\sum_{k=1}^{N-j-1}p_k\right)^i\right)$$

We claim deviating to the following strategy is profitable: S_1' where $a_N' = a_N$ is played with probability $p_N - \epsilon$ and $a_j' = a_j + \eta$ is played with probability $p_j + \epsilon_j$, for $j \neq N$, where $\sum_j^{N-1} \epsilon_j = \epsilon$ (Again, $\epsilon, \eta, \epsilon_j > 0\ \forall j$).[1] The expected payoff to firm 1 playing strategy S_1' is

$$\begin{aligned} u_1(S_1', S_{-1}) = &\sum_{j=0}^{N-1}\left(\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{1}{n-i}p_{N-j}^{n-i}\left(\sum_{k=1}^{N-j}p_k\right)\right) \\ &-\epsilon\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{1}{n-i}p_N^{n-i-1}(1-p_N)^i \\ &+\sum_{j=1}^{N-1}\left(\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{n-i-1}{n-i}p_{N-j}^{n-i}\left(\sum_{k=1}^{N-j}p_k\right)^i\right) \end{aligned}$$

Note that the deviation is profitable for firm 1 if

$$\epsilon\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{1}{n-i}p_N^{n-i-1}(1-p_N)^i < \sum_{j=1}^{N-1}\left(\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{n-i-1}{n-i}p_{N-j}^{n-i}\left(\sum_{k=1}^{N-j}p_k\right)^i\right),$$

which holds for a sufficiently small vector $(\epsilon_1, \dots, \epsilon_{N-1})$.

Second, we extend this argument to show that there can be no distributions with point masses on any point in $[0,1)$. Using an analogous argument to that used in Lemma 1.1, it is easy to see that there cannot be multiple point masses. Accordingly, it remains to show that there cannot be a single point mass. We will show that there cannot be an atom at any point $b \in (0,1)$: suppose for the sake of contradiction that that there is a symmetric equilibrium where each firm plays a point mass of size p on point b. That is, each firm plays strategy S that consists of a distribution F and a point mass of size p on point b. Let $H(x) = F^{n-1}$. Then, firm 1's payoff is

$$u_1(S_1, S_{-1}) = \int_0^1\int_0^y h(x)f(y)dxdy + p\sum_{i=0}^{n-1}\binom{n-1}{i}\left(\frac{1}{n-i}\right)F(b)^i p^{n-1-i}$$

[1]Note that we can always find such an $\eta > 0$.

Next, let firm 1 deviate by introducing a tiny point mass of size ϵ at 0 and moving the other point mass to $b + \eta$ and reducing its size slightly to $p - \epsilon$ ($\epsilon, \eta > 0$); call this strategy $S_1^{'}$. The payoff to firm 1 is

$$u_1(S_1^{'}, S_{-1}) = \int_0^1 \int_0^y h(x)f(y)dxdy + (p-\epsilon)\sum_{i=0}^{n-1}\binom{n-1}{i}F(b+\eta)^i p^{n-1-i}$$

Suppose that this is not a profitable deviation. This holds if and only if

$$p\sum_{i=0}^{n-1}\binom{n-1}{i}\left(\frac{1}{n-i}\right)F(b)^i p^{n-1-i} \geq (p-\epsilon)\sum_{i=0}^{n-1}\binom{n-1}{i}F(b+\eta)^i p^{n-1-i}$$

or,

$$\begin{aligned}\epsilon p^{n-1} + p\sum_{i=1}^{n-1}\binom{n-1}{i}\left(\frac{1}{n-i}\right)F(b)^i p^{n-1-i} \\ \geq \frac{n-1}{n}p^n + (p-\epsilon)\sum_{i=1}^{n-1}\binom{n-1}{i}F(b+\eta)^i p^{n-1-i}\end{aligned}$$

Clearly, as ϵ and η go to zero we achieve a contradiction. Hence, there is a profitable deviation and so this is not an equilibrium. It is clear that there cannot be an equilibrium with a point mass on 0 (since otherwise a firm could "move" its mass point slightly higher than zero for a discrete jump in its payoff) and so we omit a proof. ■

The mechanics behind this result are clear. There can be no mass points on any point other than 1 since a firm can always deviate by moving its mass point infinitesimally higher and achieving a discrete jump in its payoff. We can quickly gain some intuition for this result by thinking about the symmetric vector of strategies where each firm chooses a fully informative signal. Any firm can deviate profitably by instead putting some $\mu - \epsilon$ (ϵ small) weight on 1 and $1 - \mu + \epsilon$ weight on some strictly positive point (very) close to 0. Since ϵ is so small, this deviation will "cost" the deviator next to nothing, but since the probability that all the other firms all have a realization of 0 is strictly positive, the deviator will have secured itself a discrete jump up in its payoff.

1.1 Theorem 1.1 Proof

We prove this theorem for the case where $\mu \geq 1/n$. The remaining case, $\mu < 1/n$, is proved analogously and the doubtful/incredulous reader is directed to the working paper Hulko and Whitmeyer (2017), which contains the detailed proof.

1.1.1 Equilibrium Verification

Proof. First, we show that this is an equilibrium. Accordingly, we need to show that there can be no unilateral profitable deviation. Define W as $\max_{i \neq 1} X_i$, which, recall, has a point mass on 1. Moreover, define H as the corresponding continuous portion of the distribution of W; $H := F_i^{n-1}$:

$$H(w) = (1-a)^{n-1}\frac{x}{s}$$

Evidently, it suffices to show that our candidate strategy achieves a payoff of at least $1/n$ to the firm who uses it, irrespective of the strategy choice by the other firms. Suppose for the sake of contradiction that there is a profitable deviation, that is, firm 1 deviates profitably by playing strategy $\mathcal{G}$. Clearly, we can represent $\mathcal{G}$ as having a point mass of size c, $0 \leq c \leq \mu$ on 1 (naturally, if $c = 0$, then there is no point mass there). Written out, $\mathcal{G}$ consists of

$$G(y) \qquad \text{for } \ x \in [0, 1) \tag{$\mathcal{G}$}$$

and $\mathbb{P}(Y = 1) = c$. Define $K := \int_0^1 dG = 1 - c$. Naturally, $K \leq 1$. Then, firm 1's utility from this deviation, $u_1(\mathcal{G}, S_{-1})$, is[2]

$$\begin{aligned} u_1 &= c\frac{1-(1-a)^n}{na} + (1-a)^{n-1}\left(K - G(s)\right) + \int_0^s \int_0^y h(x)g(y)dxdy \\ &= \frac{c}{n\mu} + (1-a)^{n-1}\left(K - G(s)\right) + \int_0^s \int_0^y h(x)g(y)dxdy \end{aligned}$$

Evidently, this is a profitable deviation if and only if $u_1 > 1/n$; that is,

$$\frac{c}{n\mu} + (1-a)^{n-1}\left[K - G(s)\right] + \int_0^s \frac{(1-a)^{n-1}}{s} yg(y)dy > \frac{1}{n} \tag{1}$$

Rearranging,

$$\frac{c}{n\mu} + (1-a)^{n-1}\left[K - G(s)\right] + \int_0^s \frac{(1-a)^{n-1}}{s} yg(y)dy > \frac{1}{n}$$

which simplifies to

$$K > \int_s^1 \frac{1}{s} yg(y)dy + G(s)$$

since $\int_0^1 (1/s)yg(y)dy = \mu - c$ and $s = n\mu(1-a)^{n-1}$. It is clear that $\int_s^1 \frac{1}{s} yg(y)dy \geq \int_s^1 g(y)dy$ and thus we have

$$K > \int_s^1 \frac{1}{s} yg(y)dy + G(s) \geq \int_0^s g(y)dy + \int_s^1 g(y)dy = K$$

We have established a contradiction and thus the result is shown.

■

1.1.2 Equilibrium Uniqueness

First,

[2] Note that the first term, $c(1-(1-a)^n)/na$, is derived below, in the proof of Lemma 1.2.

Claim 1.1.

$$\mu \sum_{i=0}^{n-1} \binom{n-1}{i} \left(\frac{1}{n-i}\right) (1-a)^i a^{n-1-i} = \mu \frac{1-(1-a)^n}{na}$$

Proof. Define $k := n-1-i$, and so we have

$$\mu \sum_{i=0}^{n-1} \binom{n-1}{i} \left(\frac{1}{n-i}\right) (1-a)^i a^{n-1-i} = \mu(1-a)^{n-1} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{k+1} \left(\frac{a}{1-a}\right)^k$$

Then, we have the identity.

$$\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} w^k = \frac{(w+1)^{n+1}-1}{(n+1)w}$$

and so we simply set $w := \frac{a}{1-a}$, and after some algebra the proof is completed. ∎

Then,

Lemma 1.2. *Suppose that in a symmetric equilibrium each firm puts a point mass of size $a \geq 0$ on 1. Then, a must satisfy $a \geq \mu\left[1-(1-a)^n\right]$.*

Proof. Let each firm play strategy $S_i = S$ where they each put weight a on 1. Suppose that firm 1 deviates and plays strategy $\hat{S}_1$ consisting of random variable Y distributed with value 1 with probability μ and 0 with probability $1-\mu$.

Then, firm 1's payoff is

$$u_1(\hat{S}_1, S_{-1}) = \mu \sum_{i=0}^{n-1} \binom{n-1}{i} \left(\frac{1}{n-i}\right) (1-a)^i a^{n-1-i} \tag{2}$$

We use Claim 1.1 and write Equation 2 as

$$u_1(\hat{S}_1, S_{-1}) = \mu \frac{1-(1-a)^n}{na} \tag{3}$$

This must be less than or equal to $1/n$, or

$$a \geq \mu\left[1-(1-a)^n\right]$$

∎

There must also be a continuous portion of the distribution on some interval $[t, s]$ with $t \geq 0$, $s \leq 1$. Accordingly, our candidate equilibrium strategy, $\mathcal{F}_i$, is of the following form

$$\mathcal{F}_i = \begin{cases} 0 & x \in [0, t) \\ F_i & x \in [t, s) \\ 1-a & x = s \end{cases} \tag{$\mathcal{F}_i$}$$

with $0 \le t < s \le 1$ and $\mathbb{P}(X = 1) = a$.[3] We look for a symmetric equilibrium. Observe that distributions F_i must be such that

$$\int_t^s x f_i(x) dx = \mu - a$$

Fix F_j for $j \neq i$ and define H as F_j^{n-1}. Given this distribution, we have the necessary condition that F_i maximizes

$$\frac{1 - (1-a)^n}{n} + \int_t^s f_i(x) H(x) dx$$

Next, we define the functional $J[f]$:

$$J[f_i] = \int_t^s f_i(x)H(x)dx - \lambda_0 \left[\int_t^s f_i(x)dx - (1-a) \right] - \lambda \left[\int_t^s x f_i(x) dx - \mu + a \right]$$

and take the functional derivative:

$$\frac{\delta J(f(x))}{\delta f(x)} = H(x) - \lambda_0 - \lambda x$$

This must equal 0 at a maximum, so we have

$$H(x) = \lambda_0 + \lambda x$$

Then, by symmetry, $H(\cdot) = F_i^{n-1}(\cdot)$. Moreover, we have two initial conditions that allow us to obtain t and s. Using the conditions $F_i(t) = 0$ and $F_i(s) = (1-a)$, the equilibrium distribution, F_i, must be

$$F_i(x) = (1-a) \left(\frac{x-t}{s-t} \right)^{1/(n-1)}$$

Note that we also need $\int_t^s x f_i(x) dx = \mu - a$, which reduces to

$$a = \frac{n\mu - \left[s + (n-1)t \right]}{n - \left[s + (n-1)t \right]} \tag{4}$$

We finish my showing that t must be 0 and by pinning down the size of a.

Lemma 1.3. *The lower bound of the continuous portion of the distribution, t, must be* 0.

Proof. Let firms 2 through n play F_i supported on $[t, s]$ and have a point mass of size a on 1. Suppose for the sake of contradiction that $t > 0$. Recall,

$$F_i(x) = (1-a) \left(\frac{x-t}{s-t} \right)^{1/(n-1)}$$

Thus, the cdf of the maximum of this, $H := F^{n-1}$, is

$$H(x) = (1-a)^{n-1} \left(\frac{x-t}{s-t} \right)$$

[3]It is clear that the weight on 1, a, cannot be μ in the symmetric equilibrium. To see this, note that such a value for a would beget a distribution with binary support, which we already ruled out in Lemma 1.1.

Let firm 1 play some alternate strategy G supported on $[0, s]$ such that the density is positive on some portion of $[0, t]$ and have a point mass of size a on 1. Then, firm 1's expected payoff is:

$$
\begin{aligned}
u &= \frac{1-(1-a)^n}{n} + \int_t^s \int_t^y h(x)g(y)dxdy \\
&= \frac{1-(1-a)^n}{n} + \int_t^s (1-a)^{n-1}\left(\frac{y}{s-t}\right)g(y)dy - \int_t^s (1-a)^{n-1}\left(\frac{t}{s-t}\right)g(y)dy \\
&= \frac{1}{n} + \int_0^t (1-a)^{n-1}\left(\frac{t-y}{s-t}\right)g(y)dy > \frac{1}{n}
\end{aligned}
$$

where we used,

$$
\begin{aligned}
&\frac{1-(1-a)^n}{n} + \int_0^s (1-a)^{n-1}\left(\frac{y}{s-t}\right)g(y)dy - \int_0^s (1-a)^{n-1}\left(\frac{t}{s-t}\right)g(y)dy \\
&= \frac{1-(1-a)^n}{n} + (1-a)^{n-1}\left(\frac{\mu-a}{s-t}\right) - (1-a)^n\left(\frac{t}{s-t}\right)
\end{aligned}
$$

Thus, there is a profitable deviation and so t must equal 0. ■

Lemma 1.4. *The weight on* 1*,* a*, is given by* $a = \mu - \mu(1-a)^n$*.*

Proof. Recall that in Lemma 1.2 we show that $a \geq \mu - \mu(1-a)^n$. Thus, it is sufficient to show here that $a \leq \mu - \mu(1-a)^n$.

We divide the following into two cases. In the first case, suppose that $\mu \leq s$. Note that it cannot be a profitable deviation for a firm to play a strategy consisting of s played with probability μ/s and 0 with probability $1-\mu/s$. This condition is equivalent to Inequality 5:

$$\frac{1}{n} \geq \frac{(1-a)^{n-1}\mu}{s} \tag{5}$$

Or,

$$s \geq n(1-a)^{n-1}\mu \tag{6}$$

From Equation 4, and using the fact that $t = 0$, we have

$$s = \frac{n(\mu-a)}{(1-a)} \tag{7}$$

We substitute this into inequality 6 and obtain

$$
\begin{aligned}
\frac{n(\mu-a)}{(1-a)} &\geq n(1-a)^{n-1}\mu \\
\mu - \mu(1-a)^n &\geq a
\end{aligned}
$$

For the second case, suppose now that $\mu > s$. By similar logic to the above, it cannot be a profitable deviation for a firm to play a strategy consisting of s played with probability $(1-\mu)/(1-s)$ and 1 with probability $(\mu-s)/(1-s)$. That is,

$$\frac{1}{n} \geq \left(\frac{1-\mu}{1-s}\right)(1-a)^{n-1} + \left(\frac{\mu-s}{1-s}\right)\left(\frac{1-(1-a)^n}{na}\right) \tag{8}$$

Suppose for the sake of contradiction that $a > \mu-\mu(1-a)^n$. Additionally, for convenience, define $k := (1-a)^n$. Inequality 8 can be rearranged to obtain:

$$s\,(1-k-a) \geq an(1-\mu)\frac{k}{(1-a)} + \mu(1-k) - a$$

We substitute in Equation 7 and rearrange to obtain:

$$\begin{aligned} n(\mu-a)\,(1-k-a) &\geq ank(1-\mu) + \mu(1-k)(1-a) - a(1-a) \\ \mu\,(n+a-an-1) - (1-a)a(n-1) &\geq \mu k\,((n-1)(1-a)) \end{aligned} \tag{9}$$

Our assumption above that $a > \mu-\mu(1-a)^n$ is equivalent to $\mu k > \mu - a$. We substitute this into Inequality 9 and cancel:

$$\begin{gathered} \mu\,(n+a-an-1) - (1-a)a(n-1) > (\mu-a)\,((n-1)(1-a)) \\ 0 > 0 \end{gathered}$$

We have achieved a contradiction and have thereby shown that $a \geq \mu-\mu(1-a)^n$. This, combined with Lemma 1.2 allows us to conclude the result, that $a = \mu-\mu(1-a)^n$.

■

1.2 Corollary 1.1 Proof

Proof. For convenience, define $b = 1-a$, and recall (see Theorem 1.1) that we have

$$\mu = \frac{1-b}{1-b^n}$$

Define the right hand side of this expression as φ. For $b \in [0,1]$, φ is decreasing in b and therefore increasing in a over the same interval. Moreover,

$$\frac{\partial \varphi}{\partial n} = \frac{(1-b)b^n \ln(b)}{(b^n-1)^2} < 0$$

Thus, as n increases, the a needed to satisfy the above expression must increase. That is, more and more weight is put on 1. Concurrently, s, or the upper bound of the continuous portion of the distribution is shrinking, since, (as shown in the proof of Theorem 1.1)

$$s = \frac{n(\mu-a)}{1-a}$$

and thus

$$\frac{\partial s}{\partial a} = \frac{-n(1-\mu)}{(1-a)^2}$$

Furthermore, as n goes to infinity, we see that a goes to μ.

■

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