Mathematics Magazine

ISSN: 0025-570X (Print) 1930-0980 (Online) Journal homepage: https://www.tandfonline.com/loi/umma20

A Game of Nontransitive Dice

Artem Hulko & Mark Whitmeyer

To cite this article: Artem Hulko & Mark Whitmeyer (2019) A Game of Nontransitive Dice, Mathematics Magazine, 92:5, 368-373, DOI: 10.1080/0025570X.2019.1662263

To link to this article: https://doi.org/10.1080/0025570X.2019.1662263

Published online: 20 Dec 2019.

A Game of Nontransitive Dice

ARTEM HULKO
Tusculum University
Greeneville, TN 37745
artemhulko@gmail.com

MARK WHITMEYER
University of Texas at Austin
Austin, TX 78712
mark.whitmeyer@utexas.edu

If Hercules and Lychas play at dice
Which is the better man, the greater throw
May turn by fortune from the weaker hand.

William Shakespeare,
The Merchant of Venice

Sets of nontransitive dice are fascinating mathematical objects that have attracted the curiosity of many for nearly fifty years. They first came into the limelight in one of Martin Gardner's column [8] and are one of a larger class of nontransitivity "paradoxes" (see [2, 16]), which also include the well-known Condorcet voting paradox, as described in [7].

The past few years have seen a surge in interest in the topic, including [1, 4, 15]. The underlying ideas have not been limited merely to dice; one notable work is [10], which instead reinterprets the scenario through throws of unfair *coins*. Nontransitive dice have even been the subject of investigation by the well-known polymath project (see [12]), and indeed we borrow some terminology from that paper.

When one speaks of dice, one usually speaks also of a game, and when a game is the topic at hand, a natural question is how it should be *played*. Along those lines, several papers have investigated how nontransitive dice should be thrown in a strategic interaction of two or more players. Rump [14] was the first one to do so: he explores a two-player game in which each player may choose one of the four six-sided Efron dice and finds the set of equilibria, before extending the analysis to cover the situation in which each player chooses two such dice.

Here, we investigate a broader problem: we consider a two-player, simultaneous-move game in which each player selects a general n-sided die and throws it. The player with the highest face showing wins a reward, normalized to 1, and each player receives $1/2$ in the event of a tie. For our game, we use the Nash equilibrium solution concept. Note that this game is a constant-sum game; therefore it is equivalent to a zero-sum game for which a Nash equilibrium is a saddle point. We show that, for $n > 3$, there is a single, unique, pure-strategy Nash equilibrium in which both players play the *standard* n-sided die where each possible value, $1, 2, \dots, n$, occurs with probability $1/n$. There may be additional mixed strategy equilibria; however, in this analysis we focus exclusively on pure strategy equilibria and henceforth, by Nash equilibrium or equilibrium, we mean only those in pure strategies.

Moreover, our proof of uniqueness is constructive and contains an algorithm that, for any nonstandard die, generates a die that beats it. We introduce the idea of a *one-step die*: a die that is the result of a simple modification of the standard die in which one dot is moved from one face to another. Intuitively, such a die is merely "one-step"

Math. Mag. **92** (2019) 368–373. doi:10.1080/0025570X.2018.1662263
MSC: Primary 91A05, Secondary 91A60; 60C05

away from the standard die. We show that for any nonstandard die, there is always at least one one-step die that beats it.

Two additional papers bear mention. The closest paper to this one, [6], considers the same problem, where for some fixed integer n, two players each choose a die and throw against each other. The authors show that the standard die ties every other die, and that every nonstandard die loses to some other die. Another paper, [5], also explores dice games though in a slightly more general setting, and the existence and uniqueness of an equilibrium in which both players each throw the standard die follows from their Propositions 6 and 8.

Our paper differs from [5,6] in the following key ways. We provide different proofs of the existence and uniqueness of the Nash equilibrium in the game, and we are able to do so exclusively using elementary mathematics. Additionally, our proof is constructive and we formulate a simple algorithm that allows us, for any nonstandard die, to generate a die that beats it. Moreover, our last result—that for any nonstandard die, there is a one-step die that beats it—is also novel.

Finally, dice games can be placed in a more general context, as a member of the family of Colonel Blotto games. First developed by E. Borel in 1921 (see [3]), a burgeoning literature has resulted, due to the game's general applications in economics, operations research, political science, and other areas. Some recent papers include [9, 13]. In another paper [11], we explore an n-player continuous version of this game played on the interval [0, 1], which is then extended in [17] to a dynamic setting. For two players, the unique equilibrium is the continuous analog of the unique equilibrium here, the uniform distribution.

The basic game

Fix a positive integer n and define an n-tuple $D = (d_1, d_2, \ldots, d_n)$ with $1 \leq d_1 \leq d_2 \leq \cdots \leq d_n \leq n$ such that $\sum d_i = n(n+1)/2$. We then define a general n-sided die (henceforth just a "die") as a random variable, D, that takes values in the finite set $\{1, 2, \ldots, n\}$, provided the distribution satisfies the following conditions:

1. For each $i = 1, 2, \ldots, n$, the probability that a certain value occurs, $\mathbb{P}[D = i] = d_i$, is a multiple of $1/n$.
2. The expectation of the random variable is $\mathbb{E}[D] = \sum_{i=1}^{n} i \cdot d_i = \frac{n+1}{2}$.

Denote the set of all n-sided dice by $\mathcal{D}_n$. As mentioned above, the standard n-sided die S is the die where each possible value occurs with probability $1/n$.

Example 1. Five 4-sided dice appear in the set $\mathcal{D}_4$:

$$\mathcal{D}_4 = \{[1, 1, 4, 4], [2, 2, 2, 4], [1, 3, 3, 3], [2, 2, 3, 3], S = [1, 2, 3, 4]\}.$$

The game Two players, Amy and Bob, play the following one shot game for fixed n. Amy and Bob each independently select and roll any n-sided die, $A, B \in \mathcal{D}_n$. The payoff to a player is the expected gain, where the reward is 1 for throwing the higher number, 1/2 for a tie, and 0 for a lower number. Amy's expected payoff is the probability that the realization of her throw is higher than the realization of Bob's throw (with ties settled by a fair coin flip), and Bob's payoff is Amy's mirror.

A *strategy* for Amy (and analogously for Bob) is simply a choice of die $A \in \mathcal{D}_n$. For any pair of strategies, (A, B), Amy's expected payoff and Bob's expected payoff, are, respectively,

$$U_{\text{Amy}}(A, B) = \Pr(A > B) + (1/2)\Pr(A = B) \text{ and}$$
$$U_{\text{Bob}}(A, B) = \Pr(B > A) + (1/2)\Pr(A = B),$$

where $U_{\text{Amy}}(A, B) + U_{\text{Bob}}(A, B) = 1$ and $U_{\text{Amy}}(B, A) = U_{\text{Bob}}(A, B)$.

Example 2. Suppose Amy and Bob choose dice $A = [1, 1, 4, 4]$ and $B = [2, 2, 2, 4]$ from Example 1, respectively. Then $U_{\text{Amy}}(A, B) = 7/16$ and $U_{\text{Bob}}(A, B) = 9/16$.

The main result of this paper is the following theorem.

Theorem. *For any n, the unique Nash equilibrium of the two-player game is where both players play the standard die S.*

We prove this theorem through two propositions by first showing that the strategy pair (S, S) is a Nash equilibrium, and then, in the case where $n \geq 4$, by proving that (S, S) is the unique equilibrium. Before we proceed, we remind the reader of the definition of a Nash equilibrium.

Definition. A pair of strategies (A, B) is a (pure strategy) Nash equilibrium if neither player, holding his or her opponent's strategy fixed, has a profitable deviation to any other strategy. Formally, (A, B) is a pure strategy Nash equilibrium if $U_{\text{Amy}}(A, B) \geq U_{\text{Amy}}(A', B)$ for all $A' \in \mathcal{D}_n$, and $U_{\text{Bob}}(A, B) \geq U_{\text{Bob}}(A, B')$ for all $B' \in \mathcal{D}_n$.

Now, our first proposition.

Proposition 1. *The strategy pair (S, S) is a Nash equilibrium.*

Proof. We begin by showing that for either $i \in \{\text{Amy, Bob}\}$ and for all $D \in \mathcal{D}_n$, $U_i(S, D) = U_i(D, S) = 1/2$.

Note that it suffices to prove that $U_{\text{Amy}}(D, S) = 1/2$, since that clearly implies $U_{\text{Bob}}(S, D) = 1/2$. Suppose that Bob chooses the standard die S and that Amy chooses an arbitrary die D. If the realization of D is d_i (i.e., when D is rolled and "lands" showing face d_i), then with probability $(d_i - 1)/n$, D beats the standard die, and with probability $1/n$, D ties the standard die. Hence,

$$U_{\text{Amy}}(D, S) = \sum_{i=1}^{n} \left(\frac{1}{n}\right)\left(\frac{d_i - 1}{n} + \frac{1}{n} \cdot \frac{1}{2}\right) = \left(\frac{1}{n}\right)\left(\mathbb{E}[D] - \frac{1}{2}\right) = \frac{1}{2}.$$

Since (S, S) gives Amy a payoff of $1/2$, she cannot profit by deviating from S. ∎

It remains to show uniqueness, which we accomplish in the following proposition. For any two dice $A, B \in \mathcal{D}_n$, we say that A *beats* B if the number of pairs (a_i, b_j) with $a_i > b_j$ exceeds the number of pairs with $a_i < b_j$.

Proposition 2. *The Nash equilibrium (S, S) is unique for $n \geq 4$.*

Proof. Clearly, for any strategy pair, player i has a profitable deviation if and only if there is a die that beats her opponents die. Our proof is constructive and we show that for any die $B \neq S$, we can construct a die, G, that beats it.

To that end, let $B = [b_1, b_2, \ldots, b_n]$ and recall that $S = [1, 2, \ldots, n]$. For $k = 1, 2, \ldots, n$, define γ_k by $\gamma_k = |\{b_i | b_i = k\}|$. By construction of the dice,

$$\sum_{k=1}^{n} \gamma_k = n \quad \text{and} \quad \sum_{k=1}^{n} k\gamma_k = \frac{n(n+1)}{2}.$$

Next, for $k = 1, 2, \ldots, n-1$, define ξ_k as $\xi_k = \gamma_k + \gamma_{k+1}$. To construct a die G that beats B, we need simply find a pair (ξ_i, ξ_j) with $\xi_i > \xi_j$ (and so clearly $j \neq i$) and $i \neq j+1$. To see this, we take a look at what happens when we match the standard die with a die D represented by $(\gamma_1, \gamma_2, \ldots, \gamma_n)$. Face i of the standard die defeats $\gamma_1 + \cdots + \gamma_{i-1}$ faces of D and loses to $\gamma_{i+1} + \cdots + \gamma_n$ faces of D.

Observe what happens when we move a dot from face $j+1$ to face i on the standard die: the number of wins changes by $\gamma_i - \gamma_j$ and the number of losses changes by $\gamma_{j+1} - \gamma_{i+1}$. This new die is a one-step die, and for this die to dominate D, we need $\gamma_i - \gamma_j > \gamma_{j+1} - \gamma_{i+1}$; equivalently, $\gamma_i + \gamma_{i+1} > \gamma_j + \gamma_{j+1}$. Before we return to the proof, let's consider an example, and then we prove a necessary lemma.

Example 3. Suppose player A chooses die X from our previous examples, $X = [1, 1, 4, 4]$. We have $\gamma_1 = 2$, $\gamma_2 = \gamma_3 = 0$, and $\gamma_4 = 2$, and so $\xi_1 = 2$, $\xi_2 = 0$, and $\xi_3 = 2$. Evidently, $\xi_1 > \xi_2$ and $1 = i \neq j + 1 = 2 + 1 = 3$. Hence, adjusting the standard die S, we can add 1 to s_1 and subtract 1 from s_3 to yield the die $Y = [2, 2, 2, 4]$, which is a one-step die that beats X. Indeed, should player B choose Y she would achieve a payoff of $9/16 > 1/2$.

If $\xi_a \neq \xi_b$ for some a, b, then there must be some i, j with $\xi_i > \xi_j$. Thus, we establish the following lemma:

Lemma. *If $n \geq 4$, then for any nonstandard n-sided die there exists a pair $a, b \in \{1, 2, \ldots, n\}$, for which $\xi_a \neq \xi_b$.*

Proof. The equality $\xi_a = \xi_b$ holds for all $a, b \in \{1, 2, \ldots, n\}$ if and only if

$$\gamma_1 + \gamma_2 = \gamma_2 + \gamma_3 = \gamma_3 + \gamma_4 = \cdots = \gamma_{n-1} + \gamma_n$$

which holds if and only if

$$\begin{aligned} &\gamma_1 = \gamma_3 = \cdots = \gamma_k \quad \text{for all odd integers} \quad k \in \{1, 2, \ldots, n\} \text{ and} \\ &\gamma_2 = \gamma_4 = \cdots = \gamma_j \quad \text{for all even integers} \quad j \in \{1, 2, \ldots, n\}. \end{aligned} \tag{1}$$

We also have the following two relationships:

$$\sum_{k \text{ odd}}^{n} \gamma_k + \sum_{j \text{ even}}^{n} \gamma_j = n \tag{2}$$

and

$$\sum_{k \text{ odd}}^{n} k\gamma_k + \sum_{j \text{ even}}^{n} j\gamma_j = \frac{n(n+1)}{2}. \tag{3}$$

Note that any nonstandard die must have some $\gamma_i = 0$. Then either $\gamma_i = 0$ for all odd i or $\gamma_i = 0$ for all even i.

Suppose n is odd and that $\gamma_1 = 0$. From equation (2) we have $(n-1)\gamma_2 = 2n$, which does not have a solution in integers n, γ_2 for $n > 3$. Next, suppose n is odd and that $\gamma_2 = 0$. From equation (2) we have $(n+1)\gamma_1 = 2n$, which does not have a solution in integers n, γ_1 for $n > 1$. Thus, we conclude that n cannot be odd.

Suppose n is even and that $\gamma_1 = 0$. From equations (1) and (2) we must have $\gamma_2 = 2$, and from equations (1) and (3) we get that $n + 2 = 2(n+1)$, which is obviously a contradiction. Finally, suppose n is even and that $\gamma_2 = 0$. From equations (1) and (2) we must have $\gamma_1 = 2$, and from equations (1) and (3) we must have that $2(n+1) = n+1$, which is also a contradiction. Thus, we have proved the lemma. ■

To wrap up the proof of Proposition 2 we need to verify that we cannot have the situation in which the only pair ξ_i, ξ_j that satisfies $\xi_i > \xi_j$ occurs when $i = j + 1$. To that end, suppose $\xi_i > \xi_j$ for $i = j + 1$. First, let $j \neq 1$. Then, if $\xi_{j-1} \leq \xi_j$, relabel $j - 1$ as j', which yields $\xi_i > \xi_{j'}$ for $i \neq j' + 1$. On the other hand, if $\xi_{j-1} > \xi_j$, relabel $j - 1$ as i', implying $\xi_{i'} > \xi_j$ for $i' \neq j + 1$. Next, let $j = 1$. If $\xi_{i+1} \geq \xi_i$, relabel $i + 1$ as i', which yields $\xi_{i'} > \xi_j$ for $i' \neq j + 1$. If, instead, $\xi_{i+1} < \xi_i$, relabel $i + 1$ as j', and thus we have $\xi_i > \xi_{j'}$ for $i \neq j' + 1$. ■

We may also write the following corollary, which we have proved along the way.

Corollary. *Let $n \geq 4$. Then, for any die $B \neq S$, there exists a one-step die G that beats B.*

Note that given some die $B \neq S$, the algorithm developed in our proof yields *every* winning one-step die (i.e., a one-step die that beats B). Moreover, it is easy to see how by "flipping" the algorithm we could also obtain the set of losing one-step dice. Finally, the algorithm also enables us to find the "best" (and "worst") one-step dice to play versus B: the die (or dice) that have the greatest (or least) chance of beating B.

This last result is somewhat surprising. One natural notion of "closeness" of dice is that two dice D and D' are close if the dice are one step away from each other. That is, if we could move a dot from a particular face on dice D to another face and thereby obtain die D' (and vice-versa). Hence, the corollary can be interpreted as saying that for any nonstandard die, there is a die close to the standard die that beats it.

REFERENCES

[1] Angel, L., Davis, M. (2016). A direct construction of non-transitive dice sets. ArXiv e-prints 1610.08595. arxiv.org/pdf/1610.08595

[2] Blyth, C. (1972). Some probability paradoxes in choice from among random alternatives. *J. Am. Stat. Assoc.* 67: 366–373.

[3] Borel, E. (1921). La théorie du jeu et les équations integrales à noyau symétrique. *C. R. Acad. Sci.* 173: 1304–1308; English translation by Savage, L. (1953). The theory of play and integral equations with skew symmetric kernels. *Econometrica*. 21: 97–100.

[4] Conrey, B., Gabbard, J., Grant, K., Liu, A., Morrison, K. (2016). Intransitive dice. *Math. Mag*. 89(2): 133–143.

[5] De Schuymer, B., De Meyer, H., De Baets, B. (2006). Optimal strategies for equal-sum dice games. *Discrete Appl. Math*. 154: 2565–2576.

[6] Finkelstein, M., Thorp, E. O. (2007). Nontransitive dice with equal means. In: Ethier, S. N., Eadington, W. R., eds. *Optimal Play: Mathematical Studies of Games and Gambling*. Reno: Institute for the Study of Gambling and Commercial Gaming. www.math.uci.edu/ mfinkels/dice9.pdf

[7] Fishburn, P. C., Brams, S. J. (1983). Paradoxes of preferential voting. *Math. Mag*. 56(4): 207–214.

[8] Gardner, M. (1970). The paradox of the nontransitive dice. *Sci. Amer*. 223: 110–111.

[9] Hart, S. (2008). Discrete Colonel Blotto and General Lotto games. *Int. J. Game Theory*. 36(3–4): 441–460.

[10] Hetyei, G. (2016). Efron's coins and the linial arrangement. *Discrete Math*. 339(12): 2998–3004.

[11] Hulko, A., Whitmeyer, M. (2017). A game of random variables. *Mimeo*. arxiv.org/abs/1712.08716

[12] Polymath, D. H. J. (2017). The probability that a random triple of dice is transitive. *Mimeo*. gowers.files.wordpress.com/2017/07/polymath131.pdf

[13] Roberson, B. (2006). The Colonel Blotto game. *Econ. Theory*. 29(1): 1–24.

[14] Rump, C. (2001). Strategies for rolling the Efron dice. *Math. Mag*. 74(3): 212–216.

[15] Savage, R. P., Jr. (1994). The paradox of nontransitive dice. *Amer. Math. Monthly*. 101(5): 429–436.

[16] Tenney, R. L., Foster, C. C. (1976). Non-transitive dominance. *Math. Mag*. 49(3): 115–120.

[17] Whitmeyer, M. (2018). A game of martingales. *Mimeo*. arxiv.org/abs/1811.11664

Summary. We consider a two-player, simultaneous-move game where each player selects any permissible n-sided die for a fixed integer n. For any $n > 3$, there is a unique Nash equilibrium in pure strategies in which each player throws the standard n-sided die. Our proof of uniqueness is constructive, and we introduce an algorithm with which, for any nonstandard die, we can generate another die that beats it. For any nonstandard die there exists a one-step die—a die that is obtained by transferring one dot from one side to another on the standard die—that beats it.

ARTEM HULKO (MR Author ID: 1223837) is an assistant professor of Mathematics at Tusculum University. He received his B.A. from USC Upstate and his Ph.D. from UNC Charlotte. His primary areas of research are spectral analysis, scattering theory, and game theory.

MARK WHITMEYER (MR Author ID: 1236225) received his B.A. from Lehigh University and is a Ph.D. Candidate in Economics at UT Austin. He is primarily interested in game theory and information economics.

Prime Time

Try all the tactics but you'll always find the gap of one,
Whenever you increase the gap, Our appearance goes to none.
We are very basic but you know little about us
Do you really think you can explore the empire that belongs to us?
People out there know our values and worth,
Like Ramanujan, Mersenne, Fermat and Aryabhata.
Weakness in calculation had supported Fermat falsification,
The invention of contraption produces Prime size competition.
We are unbreakable because of our indivisible quality.
It's promising you, That's why your security is our guarantee.
How you count our existence between two numbers each?
We can typify any number if some of us planned to stitch.
Our counting has been building the theory of numbers,
Thus Kronecker told number theorists are like lotus eaters.
Try all the tactics but you'll always find the gap of one,
Whenever you increase the gap, Our appearance goes to none.

—Submitted by Shashi Kant Pandey
University of Delhi, India

Math. Mag. **92** (2019) 373. doi:10.1080/0025570X.2019.1445931